On Generalized Hopf Differentials

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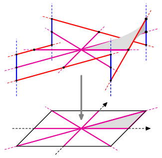

Joint work with: Harold Rosenberg (Univ. Paris VII)

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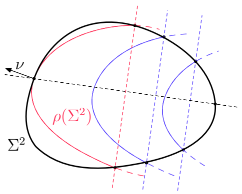
1.1. Alexandrov's Theorem

Theorem ([Alexandrov, 1955])
Let Σ^2 be a closed embedded cmc surface in $\mathbb{R}^3$, in $\mathbb{H}^3$, or in a hemi-sphere $\mathbb{S}^3_+$. Then Σ^2 is a distance sphere.



Idea of Proof.
Consider reflections through a family of (parallel) inward moving planes.
By the maximum principle, $\Sigma^2 = \rho(\Sigma^2)$ upon first contact.
Thus $\Sigma^2 = \rho(\Sigma^2)$ for all reflections ρ preserving the center of Σ^2. □

Remarks
- i) Each distance sphere $S^2 \subset \mathbb{S}^3$ is contained in a closed hemi-sphere.
- ii) In $\mathbb{S}^3$ there are Clifford tori and many other cmc surfaces of higher genus [cf. Kapouleas, 1997].

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1.2. Hopf's Theorem

Theorem ([Hopf, 1956])
Let S^2 be an immersed cmc sphere in $\mathbb{R}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$. Then S^2 is a distance sphere.

Ingredients.
- i) The Codazzi equations for $h_\Sigma = \langle\,.\,, A\,.\,\rangle$ imply: on any immersed cmc surface, $Q_H := \pi_{2,0}(h_\Sigma)$ is a holomorphic quadratic differential.
- ii) $\{\text{hol. quad. differentials on } \mathbb{S}^2 = \mathbb{CP}^1\} = 0$, hence:
$$h_\Sigma - \tfrac{1}{2}\,\mathrm{tr}(A)\,g \;=\; 2\,\mathfrak{Re}\,Q_H \;=\; 0\,.$$
- iii) Complete, totally-umbilical surfaces Σ^2 in $\mathbb{R}^3$, in $\mathbb{H}^3$, or in $\mathbb{S}^3$ are distance spheres. □

Remark
The identity $\bar\partial Q_H = 0$ can be understood as a first integral of the cmc equation.

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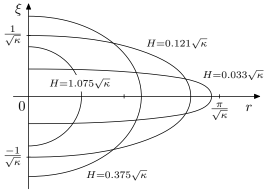
2.1. Rotationally-Invariant Cmc Spheres in $M^2_\kappa \times \mathbb{R}$

These spheres will serve as model surfaces later on!

Construction of $S^2_H \hookrightarrow M^2_\kappa \times \mathbb{R}$.



i) The relevant ODE-system:

$$\frac{\partial}{\partial s} r = -\sin\theta$$
$$\frac{\partial}{\partial s} \xi = \cos\theta$$
$$\frac{\partial}{\partial s} \theta = 2H - \cos(\theta)\,\mathrm{ct}_\kappa(r)$$

Convention: $(\cos\theta, \sin\theta)$ is the exterior unit normal vector field of the meridian curve $c(s) = (r(s), \xi(s))$.

ii) A first integral [cf. Hsiang, 1989]:

$$L := \cos(\theta)\,\mathrm{sn}_\kappa(r) - 4H\,\mathrm{sn}_\kappa(\tfrac{1}{2}r)^2$$

iii) The curve $c(s)$ intersects the fixed point set
$\iff \quad L = 0 \quad$ (or, in case $\kappa > 0$, iff $L = -4H/\kappa$).

2.1.⋆ Rotationally-Invariant Cmc Spheres in $M^2_\kappa \times \mathbb{R}$

Explicit Solution for $\kappa > 0$.

$$1 = (\kappa + 4H^2)\cdot\tfrac{1}{\kappa}\sin^2\big(\tfrac{1}{2}r\sqrt{\kappa}\big) + 4H^2\cdot\tfrac{1}{\kappa}\sinh^2\big(\tfrac{1}{2}\xi\sqrt{\kappa}\cdot\tfrac{1}{2H}\sqrt{\kappa+4H^2}\,\big)$$

Principal Curvatures.

$$h_\Sigma = \begin{pmatrix} H + \frac{\kappa}{4H}\cos^2(\theta) & 0 \\ 0 & H - \frac{\kappa}{4H}\cos^2(\theta) \end{pmatrix}$$

Remarks

i) If $0 < 4H^2 < \kappa$, the model spheres S^2_H constructed above do not project into closed hemi-spheres.

ii) The spheres S^2_H are not totally-umbilical.

iii) The bilinear forms $q := 2H\cdot h_\Sigma - \kappa\cdot d\xi^2$, however, are again multiples of the induced metric $\iota^\star g$.

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2.2. Adapting Alexandrov's Theorem

Theorem

Any closed embedded cmc surface Σ^2 in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2_+ \times \mathbb{R}$ is a rotationally-invariant vertical bigraph.

Such a bigraph Σ^2 is necessarily congruent to some S^2_H.

Idea of the Proof.

Alexandrov's moving planes argument. □

Caveats.

i) Closed embedded cmc surfaces $\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}$ that do not project into some hemi-sphere $\mathbb{S}^2_+$ are only guaranteed to be vertical bigraphs.

ii) Not all of the rotationally-invariant cmc spheres $S^2_H \hookrightarrow \mathbb{S}^2 \times \mathbb{R}$ do project into hemi-spheres.

iii) In $\mathbb{S}^2 \times \mathbb{R}$ itself, there again exist embedded cmc tori and embedded cmc surfaces of higher genus.

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2.3. What about Extending Hopf's Theorem?

Immediate Obstacles.

- For target manifolds other than space forms, the r.h.s. of the Codazzi equations does not vanish anymore:

$$\langle \nabla_X A\cdot Y - \nabla_Y A\cdot X\,, Z\rangle = \langle R(X,Y)\,\nu\,, Z\rangle = \langle X\times Y\,, G\,(\nu\times Z)\rangle\,,$$

  Here $A = \mathrm{D}\,\nu$ and $\nabla_X Y = (\mathrm{D}_X Y)^{\tan}$, and $G \equiv Ric - \frac{1}{2}Sc\cdot\mathbb{1}$ denotes the Einstein tensor.

- Conclusion: $\bar\partial Q_H \equiv \bar\partial(\pi_{2,0}(h_\Sigma)) \neq 0$.
- The model spheres S^2_H are not totally-umbilical.

Encouraging Facts.

i) The fields $q = 2H\cdot h_\Sigma - \kappa\cdot\iota^\star(d\xi^2)$ are linear combinations of h_Σ and $\iota^\star(d\xi^2)$ with constant coefficients.

ii) Their $(2,0)$-parts vanish on S^2_H, and so $Q := \pi_{2,0}(q)$ may be holomorphic on all cmc surfaces $\Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$.

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3. New Results for Cmc Spheres in $M^2_\kappa \times \mathbb{R}$

Theorem 1 ([A___ & Rosenberg, 2004])

Any cmc surface $\Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ *comes with a natural holomorphic quadratic differential given by*

$$Q \ := \ 2H \cdot \pi_{2,0}(h_\Sigma) - \kappa \cdot \pi_{2,0}(\iota^\star(\mathrm{d}\xi^2)) \ .$$

This result is proved by direct computation.
As in H. Hopf's work, Theorem 1 is the key to

Theorem 2 ([A___ & Rosenberg, 2004])

Any immersed cmc sphere $S^2 \looparrowright M^2_\kappa \times \mathbb{R}$ *is one of the rotationally-invariant model spheres* $S^2_H \hookrightarrow M^2_\kappa \times \mathbb{R}$.

Further Ingredients in the Proof of Theorem 2.

i) $\{\text{hol. quad. differentials on } \mathbb{S}^2 = \mathbb{CP}^1\} \ = \ 0$.

ii) Given $\kappa \neq 0$ and $H \in \mathbb{R}$, one can use ODE techniques in order to classify the cmc surfaces $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ with mean curvature H and with $Q \equiv 0$. □

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3.1. On the New Holomorphic Quad. Differentials

Basic Ingredients for the Proof of Theorem 1.

- On oriented surfaces $(\Sigma^2, \iota^\star g)$, the almost complex structure J is parallel, and the $\bar\partial$-operator is given by

$$\bar\partial Q(X; Y_1, Y_2) = \tfrac{1}{2}\big(\nabla_X Q + i\,\nabla_{JX} Q\big)(Y_1, Y_2) =: \nabla_{\frac{1}{2}(1+iJ)X}\, Q\,(Y_1, Y_2)\ .$$

- $A = H \cdot \mathbb{1} + A_0$, and, on surfaces, traceless symmetric endomorphisms like A_0 anti-commute with J.
- The Codazzi equations for surfaces Σ^2 in 3-manifolds:

$$\langle \nabla_X A \cdot Y - \nabla_Y A \cdot X\,, Z\,\rangle = \langle X \times Y\,, G\,(\nu \times Z)\rangle = \langle (X \times Y) \times Z\,, G\,\nu\rangle\ .$$

Here $A = \mathrm{D}\,\nu$, and G denotes the Einstein tensor.
The final expression follows, since $\nu \perp X, Y, Z$ and

$$G\,(\nu \times Z) = \mathrm{tr}(G)\cdot \nu \times Z - (G\nu) \times Z - \nu \times (GZ\,)\ .$$

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3.1.* On the New Holomorphic Quad. Differentials

Key Steps in the Proof of Theorem 1.

i) The Codazzi equations imply that

$$\bar\partial\big(\pi_{2,0}(h_\Sigma)\big)(X; Y_1, Y_2) = \langle\, \psi(X; Y_1, Y_2)\,, G\,\nu\,\rangle$$

where

$$\psi(X; Y_1, Y_2) := \tfrac{1}{2}\big[\langle X^-, Y_1^+\rangle\, Y_2^+ + \langle X^-, Y_2^+\rangle\, Y_1^+\big]\ ,$$

$$X^- := \tfrac{1}{2}(1 + iJ)\,X\ , \quad \text{and} \quad Y_\mu^+ := \tfrac{1}{2}(1 - iJ)\,Y_\mu\ .$$

ii) Computing ∇ in terms of D and ν, it follows that the vertical projectors $L := \mathrm{d}\xi^2$ satisfy

$$\bar\partial\big(\pi_{2,0}(\iota^\star L)\big)(X; Y_1, Y_2) = \langle\, Y_1^+\,, \mathrm{D}_{(X^-)}L \cdot Y_2^+\,\rangle - 2H\,\langle\, \psi(X; Y_1, Y_2)\,, L\,\nu\,\rangle$$

iii) The product structure of the targets $M^2_\kappa \times \mathbb{R}$ implies that $\mathrm{D}L = 0$ and, moreover, that $G = -\kappa\,L$. Thus

$$\bar\partial\big(\pi_{2,0}(2H \cdot h_\Sigma - \kappa \cdot \iota^\star L)\big)(X; Y_1, Y_2) \ = \ 0\ . \quad □$$

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3.1.* On the New Holomorphic Quad. Differentials

Getting some Conceptual Understanding.

i) Since $\mathrm{D}L = 0$ and $G = -\kappa\,L$, it follows from the basic structure of Wirtinger calculus that the terms $\bar\partial(\pi_{2,0}(h_\Sigma))(X; Y_1, Y_2)$ and $\bar\partial(\pi_{2,0}(\iota^\star L))(X; Y_1, Y_2)$ must both be multiples of $\langle\, \psi(X; Y_1, Y_2)\,, L\,\nu\,\rangle$. Hence there exist universal constants $a, b \in \mathbb{C}$ such that

$$\bar\partial\big(a \cdot \pi_{2,0}(h_\Sigma) - b \cdot \pi_{2,0}(\iota^\star L)\big)(X; Y_1, Y_2) \ = \ 0$$

for any immersed cmc surface $\Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$.

ii) On the rot.-invariant model spheres $S^2_H \hookrightarrow M^2_\kappa \times \mathbb{R}$, the quadratic differential $Q = \pi_{2,0}(2H \cdot h_\Sigma - \kappa \cdot \iota^\star L)$ vanishes identically, and hence $\bar\partial Q \equiv 0$, too.
This particular case now fixes the universal constants a and b above to the claimed values. □

3.2. An Auxillary Classification Result

Theorem 3 ([A___ & Rosenberg, 2004])
Let $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ be a complete immersed surface with constant mean curvature H and with $Q \equiv 0$. Suppose that $(\kappa, H) \neq 0$. Then the following holds:

- *if $4H^2 + \kappa > 0$, then Σ^2 is congruent to a rot.-inv. model sphere $S^2_H \hookrightarrow M^2_\kappa \times \mathbb{R}$.*
- *if $4H^2 + \kappa \leq 0$, then Σ^2 is a complete open surface of type D^2_H, P^2_H, or C^2_H, respectively. The three cases can be distinguished by the sign of $4H^2 + \kappa \cos^2\theta$ where $\theta := \arcsin(\mathrm{d}\xi \cdot \nu)$ denotes the Gauß angle.*

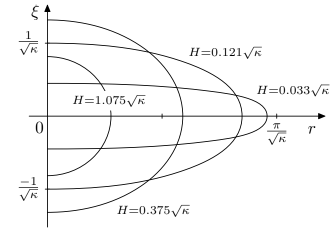
Remarks

i) Here D^2_H and C^2_H denote rotationally-inv. cmc surfaces that are homeomorphic to disks or annuli (catenoids).

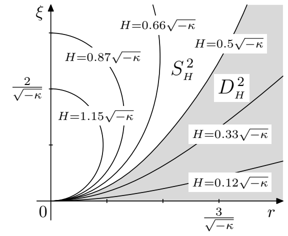
ii) The P^2_H are orbits under 2-dim. solvable subgroups $\mathsf{AN} \subset \mathsf{SO}(2,1)^+ \times \mathbb{R}$.

3.2.⋆ An Auxillary Classification Result

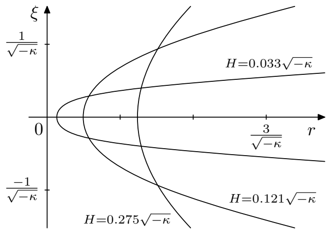
$\kappa > 0$: Meridians for S^2_H



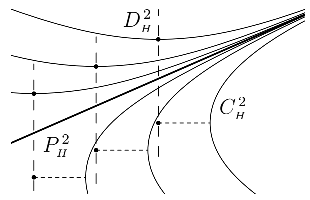
$\kappa < 0$: Meridians for S^2_H and D^2_H



$\kappa < 0$: Meridians for C^2_H



$\kappa < 0$: Meridians of P^2_H are limits



3.3. Key Steps in Proving This Classification Result

The unit normal field ν of an immersion $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ provides a lift of ι into the total space of the unit tangent bundle $\pi\colon N^5_\kappa := T_1(M^2_\kappa \times \mathbb{R}) \to M^2_\kappa \times \mathbb{R}$.

Proposition (Prolongation)
Immersed surfaces $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ with constant mean curvature H and $Q \equiv 0$ lift to integral surfaces $\nu\colon \Sigma^2 \hookrightarrow N^5_\kappa$ of an explicit 2-dimensional distribution $E_H \subset TN^5_\kappa$.

Properties of E_H.

i) E_H is invariant under the action of $\mathrm{Iso}_0(M^2_\kappa \times \mathbb{R})$. This action has 4-dim. orbits that are separated by
$$\Theta\colon N^5_\kappa \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi],$$

ii) The Gauß map $\theta\colon s \mapsto \Theta \circ c(s)$ of any meridian solves
$$\tfrac{\partial}{\partial s}\theta = \tfrac{1}{4H}\left(4H^2 + \kappa \cos^2\theta\right).$$

iii) E_H is integrable.

4. Further Generalization of the Target Spaces

Is it possible to replace the product spaces $M^2_\kappa \times \mathbb{R}$ by more general oriented Riemannian manifolds (M^3, g)?

Theorem 4 ([A___, 2006])
Let L_0 be a $\mathbb{C}$-valued, traceless, symmetric bilinear form on (M^3, g). Then the expression
$$Q := \pi_{2,0}(h_\Sigma + \iota^\star L_0)$$
defines a holomorphic quadratic differential on any surface $\iota\colon \Sigma^2 \looparrowright (M^3, g)$ with constant mean curvature H, if and only if L_0 solves the differential equation
$$\mathrm{D}_X L_0 = \tfrac{1}{2} i \left[\star X, G - 2H\, L_0\right]. \qquad (*)$$

Remark
The ODE-system $(*)$ is overdetermined. The integrability condition — even required for local solutions — imposes serious restrictions on the geometry of (M^3, g).

4.1. Results Concerning Homogeneous Bundles

Theorem 5 ([A___, 2006])

Let $(\tilde{M}^3, g)$ be a simply-connected, oriented Riemannian manifold. Then there exists a solution L_0 of

$$\mathrm{D}_X L_0 \;=\; \tfrac{1}{2} i \left[\star X, G - 2H\, L_0 \right] \,, \qquad (*)$$

iff $(\tilde{M}^3, g)$ is a hom. space with an at least 4-dimensional isometry group, or, equivalently, iff it is a space form or a homogeneous bundle $M^3_{\kappa,\tau} \to N^2_\kappa$.

New target spaces: $\mathbb{S}^3_{\mathrm{Berger}}$, $\mathsf{Nil}(3)$, and $\widetilde{\mathsf{Sl}}(2,\mathbb{R})$.

Remark

The hom. bundles $M^3_{\kappa,\tau} \to N^2_\kappa$ are the products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Berger spheres $\mathbb{S}^3_\eta$, the Heisenberg group $\mathsf{Nil}(3)$, and $\widetilde{\mathsf{Sl}}(2,\mathbb{R})$, and explicit solutions of $(*)$ are

$$L_0 := -\tfrac{\kappa - \tau^2}{2H - i\tau} \left(P - \tfrac{1}{3} \mathbb{1} \right) .$$

Here P denotes the field of vertical projectors.

4.1.* Results Concerning Homogeneous Bundles

The holomorphic quad. differentials $Q := \pi_{2,0}(h_\Sigma + \iota^\star L_0)$ that come with these solutions L_0 are the key to:

Theorem 6 ([A___, 2006])

Any immersed cmc sphere S^2 in a homogeneous bundle $M^3_{\kappa,\tau} \to N^2_\kappa$ is in fact embedded and rotationally-invariant. Thus its shape is determined by the mean curvature H.

Of couse, we have to refine Theorem 3 appropriately, too.

Remarks

i) Thus we have extended H. Hopf's result to immersed cmc spheres in homogeneous spaces representing 7 of the 8 maximal homogeneous structures [cf. Thurston].

ii) On $\mathsf{Solv}(3)$, however, the cmc equation has — due to lack of symmetry — no first integrals like our holomorphic quad. differentials. More precisely, there is no 1-dim. isotropy group.

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4.2. On the Geometry of Homogeneous 3-Manifolds

The dimension of $\mathsf{G} := \mathrm{Iso}(\tilde{M}^3, g)$ is either 3, 4, or 6. We'll discuss each case for simply-connected spaces.

a) $\dim \mathsf{G} = 6$:

These spaces have constant curvature κ. Up to scaling, they are the standard spaces $\mathbb{S}^3$, $\mathbb{R}^3$, and $\mathbb{H}^3$.
Their Einstein tensor is $G = -\kappa\, \mathbb{1}$. Evidently, $\mathrm{D}G \equiv 0$.

b) $\dim \mathsf{G} = 4$:

These spaces are homogeneous bundles $\pi_{\kappa,\tau} \colon \tilde{M}^3_{\kappa,\tau} \to \tilde{N}^2_\kappa$ over simply-connected surfaces of constant curvature κ.
They have tot.-geod. fibers and const. bundle curvature τ.

Convention: $[X, Y]^{\mathrm{vert}} = \tau \cdot X \times Y$ for all hor. vector fields.
Range of (κ, τ): the curve $\kappa = \tau^2$ must be excluded, as it yields spaces of constant curvature.

Complete list:

$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\mathbb{S}^3_{\mathrm{Berger}}$	$\mathsf{Nil}(3)$	$\widetilde{\mathsf{Sl}}(2,\mathbb{R})$

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4.2.* On the Geometry of Homogeneous 3-Manifolds

b) $\dim \mathsf{G} = 4$ (cont.):

Further properties of the spaces $\tilde{M}^3_{\kappa,\tau}$:

i) Their Einstein tensor is

$$G \;=\; -\tfrac{1}{4}\tau^2\, \mathbb{1} - (\kappa - \tau^2)\, P \,.$$

ii) Moreover, $\mathrm{D}_X G \;=\; \tfrac{1}{2}\tau\, [\star X, G]$.
So they are symmetric spaces, iff $\tau = 0$.

iii) Finally, their Cotton tensor turns out to be $-\tfrac{3}{2}\tau\, G_0$.
So they are locally conformally flat, iff $\tau = 0$.

iv) For any pair κ, τ, the isotropy group G_p of any point p is bigger than $\mathsf{SO}(2)$. It contains $\mathsf{S}(\mathsf{O}(2) \times \mathsf{O}(1))$.

v) Hence, for any horizontal geodesic $\gamma \colon \mathbb{R} \to \tilde{M}^3_{\kappa,\tau}$, there is a 180^o-rotation ϕ_γ containing γ in its fixed point set. In fact, G is generated by these rotations.

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c) $\dim \mathsf{G} = 3$:

These spaces are 3-dimensional Lie groups equipped with left-invariant metrics [cf. Milnor, 1976].

Remarks

i) There are several isomorphism classes of 3-dimen. real Lie algebras, but only one of them gives raise to a new maximal homogeneous structure: $\mathsf{Solv}(3)$.

ii) A quotient of $\mathsf{Solv}(3)$ is a torus bundle over $\mathbb{S}^1$.

iii) The geometry of $\mathsf{Solv}(3)$ is also very special:
- $\ker(Ric)$ is a 2-dim. integrable distribution. Its Weingarten map has 2 distinct eigenvalues.
- The Cotton tensor has 3 distinct eigenvalues.
- G and *Cotton* commute.

Yet, the isotropy groups are finite and, in fact, isomorphic to the dihedral group D_4.

The 4 Basic Types [cf. Figueroa, Mercuri, Pedrosa]

a) **Vertical Planes:** total preimages of straight lines, invariant w.r.t. vertical translations.

b) **Catenoids and Horizontal Umbrellas:** invariant w.r.t. a group ϕ_t of rotations around some vert. axis.

c) **Helicoids and Helicoidal Catenoids:** invariant w.r.t. a group ϕ_t of screw motions around a vert. axis.

d) **Saddle-Type Surfaces:** invariant w.r.t. a group ϕ_t of isometries that project to translations of $\mathbb{R}^2$.

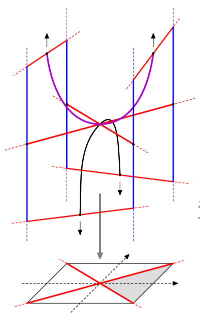
Remarks

i) The umbrellas and the saddle-type surfaces are graphs w.r.t. the Riem. submersion $\mathsf{Nil}(3) \to \mathbb{R}^2$.

ii) $Q = 0$ on umbrellas and on vertical planes, whereas $Q = \mathrm{c}\,dz^2 \neq 0$ for the saddle-type surfaces.

a) Local Scherk Surfaces.

They come as Nitsche graphs over a square in $\mathbb{R}^2$ w.r.t. the submersion $\mathsf{Nil}(3) \to \mathbb{R}^2$. Their boundary consists of the vertical geodesics over the 4 vertices of the square.



i) They are invariant w.r.t. the 180°-rotations around hor. lifts of the diagonals. (→ Schwarz reflection principle.)

ii) They do not extend to doubly-periodic minimal surfaces in $\mathsf{Nil}(3)$.

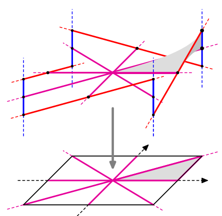
iii) Upon enlarging the square, they converge to saddle-type surfaces not umbrellas.

Application (A Weak Bernstein Theorem)

Serrin style curvature bounds for (global) minimal graphs.

b) Triply-Periodic Scherk Surfaces $\hat{\Sigma}^2$.

In order to construct these surfaces, fix a triangle $\bar{\gamma}$ in the barycentric subdivision of the fundamental square in $\mathbb{R}^2$, and proceed as follows:



i) Consider a horizontal lift of $\bar{\gamma}$ starting over the vertex of the square, and add a vertical segment to get a closed polygon γ.

ii) Solve the Plateau problem $\partial\Sigma^2 = \gamma$ and extend Σ^2 to a global minimal surface $\hat{\Sigma}^2$ by means of the Schwarz reflection principle.

Remark

$\hat{\Sigma}^2 = \Gamma \cdot \Sigma^2$ where $\Gamma \subset \mathrm{Iso}\,(\mathsf{Nil}(3))$ is the discrete subgroup generated by the four 180°-rotations around the edges of γ.

5.3. Half-Space Theorems

Theorem 7 ([A___ & Rosenberg, 2004])
Let Σ^2 be a proper, possibly branched minimal surface in the Heisenberg group Nil(3). *Suppose that Σ^2 is contained in the complement of some horizontal umbrella. Then Σ^2 is congruent to this umbrella by a vertical translation.*

Method of Proof.
The same argument as in $\mathbb{R}^3$ works, since the catenoids collapse to doubly-covered punctured umbrellas when their necksize is shrunk to 0. □

Remarks

i) There is no half-space theorem w.r.t. the level sets $\mathbb{H}^2 \times \{t_0\}$ in the product $\mathbb{H}^2 \times \mathbb{R}$.

ii) Yet, the horizontal umbrellas in Nil(3) are hyperbolic and not parabolic.

Question: Are there also half-space theorems w.r.t. the saddle-type surfaces in Nil(3) rather than the umbrellas?

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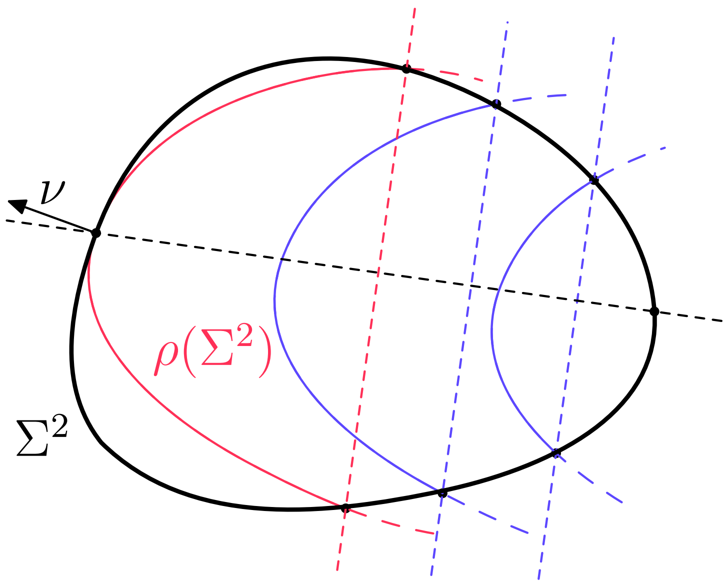
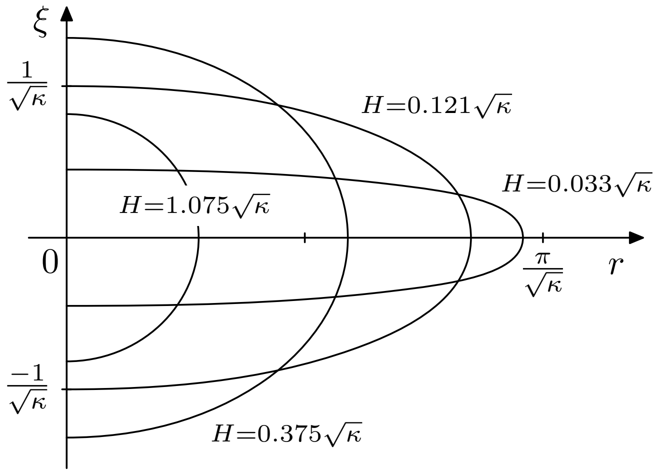
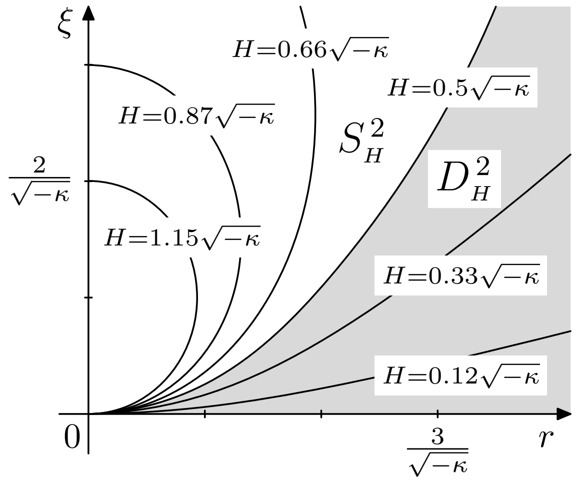
7.1. Alexandrov's Moving Planes Argument



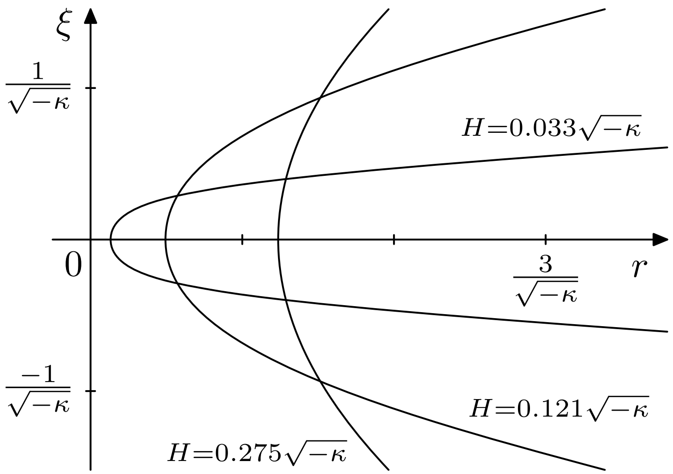
$\boldsymbol{\kappa > 0}$: Meridians for S^2_H



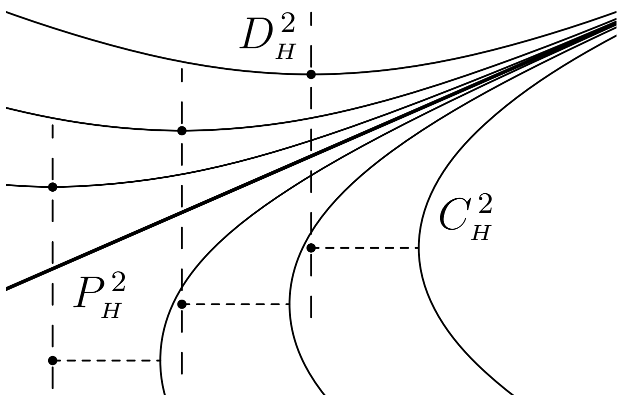
$\boldsymbol{\kappa < 0}$: Meridians for S^2_H and D^2_H



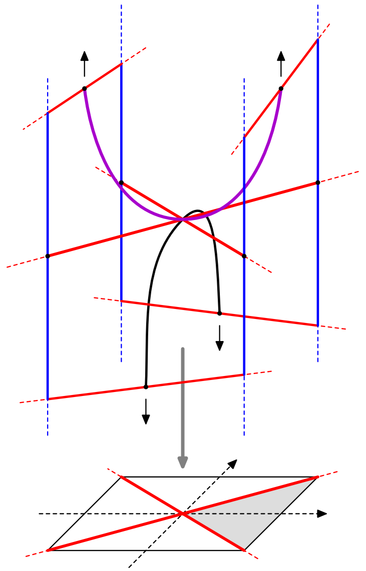
$\boldsymbol{\kappa < 0}$: Meridians for C^2_H



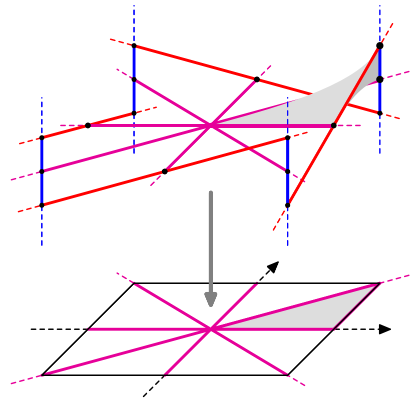
$\boldsymbol{\kappa < 0}$: Meridians of P^2_H are limits



7.3. Local Scherk Surfaces in Nil(3)



7.4. Triply-Peridic Scherk Surfaces in Nil(3)



Generalized Hopf Differentials

Uwe Abresch

References

Zoomed Figures

Alexandrov's Moving Planes Argument

Meridian Curves of the Model Surfaces

Local Scherk Surfaces in Nil(3)

Triply-Peridic Scherk Surfaces in Nil(3)