Escape from a Circle and the Riemann Hypothesis

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Phys. Rev. Lett. 94 (2005)

Dynamical Systems

$M :=$ phase space

$S^t : M \circlearrowleft$

$M \supset H :=$ "hole" (measurable subset)

disappears at the moment $t \geq 0$

H and $S^\tau \notin H$ for $0 \leq \tau < t$

ary moment of time t

rmalize measure on the

f surviving (nonescaping

y

μ_t

μ_t converges as $t \to \infty$?

New idea: Make several holes and compare the total escape rate with the sum of escape rates trough individual holes.

This information may shed some light on the dynamics of a closed (when all holes are "patched") nn zeta-function 1859

$$)\!:= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

mplex number

e Number Theorem
dre-Gauss conjecture

$$\#\{\text{primes less than } x\} \sim \int_2^x \frac{dt}{\log t}$$

Hadamard
dently and de la Vallée-Poussin

Riemann's functional equation

$$\xi(s) := \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

Euler product $\Rightarrow$ no zeros of $\zeta(s)$ with $\operatorname{Re} s > 1$

Funct-l eq-n $\Rightarrow$ no zeros of $\zeta(s)$ with $\operatorname{Re} s < 0$.

Farey numbers

rational numbers with denominators at most Q arranged in ascending order

$Q = 7$ $\quad \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}$

$\mathcal{F}_7$ $\quad \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$

$\mathrm{Card}(\mathcal{F}_Q) = N(Q)$

Franee (1924)

$$RH \Leftrightarrow \sum_{j=1} \left| \gamma_j - \frac{j}{N(Q)} \right| = O\left(Q^{1/2+\varepsilon}\right)$$

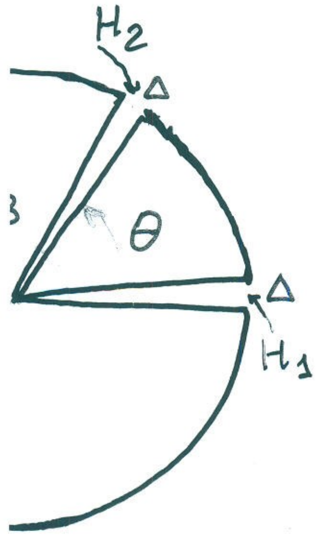
Landau (1924)

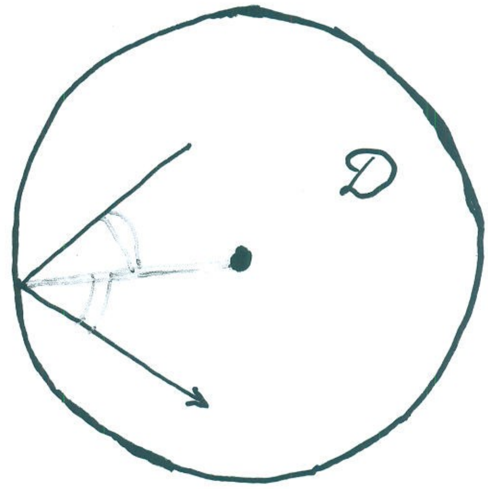
$$RH \Leftrightarrow \sum_{j=1}^{N(Q)} \left(\gamma_j - \frac{j}{N(Q)} \right)^2 = Q\left(Q^{-1+\varepsilon}\right)$$

Lagarias (2002) $\quad \sigma(n) :=$ sum of positive divisors of n

$$RH \Leftrightarrow \sigma(n) \leq H_n + \exp(H_n) \log H_n \quad \forall n$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$





1 elastic collisions from the boundary

$-\infty < t < \infty$, billiard flow.

volume in the phase space $\mathcal{M}$
served under the dynamics

iard map $T : M \to M$

$(\beta, \psi) : -\pi < \beta \le \pi, -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}\}$,

$M \to \partial D$

$\beta,\psi) = \beta, \ (\beta,\psi) \in M$

of billiard in a circle are

c if $\psi = \frac{\pi}{2} - \frac{p}{q}\pi$

ere dense if ψ is incommensurable with π

y orbits that never escape

periodic orbits that never hit $H_1 \cup H_2$

$\beta,\psi): \beta \in H_i\}, i=1,2.$ proj $\widehat{H}_i = H_i$

$(\beta_0, \psi_0) \in M$

Two (possibly overlapping) holes

$$H_1=\{\beta,\ 0<\beta<\Delta\},\ H_2=\{\beta:\theta<\beta<\theta+\Delta\}$$

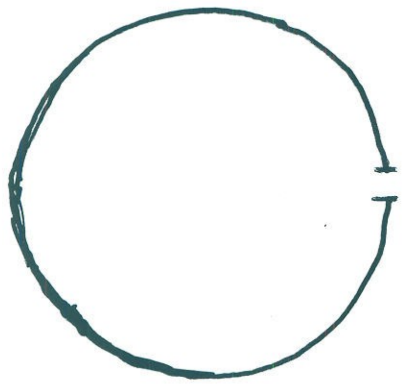
$0\le\theta<2\pi$, $\Delta>0$ at the boundary ∂D.

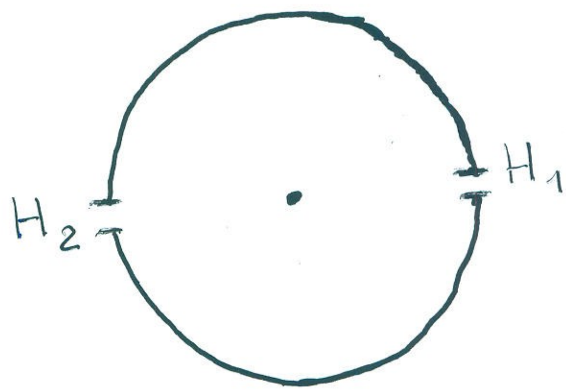
$P_1(t):=$ pr-ty of not escaping till time t when $\theta=0$
one hole

$P_2(t):=$ pr-ty of not escaping till time t when $\theta=\pi$
two opposite holes

$$RH \iff \lim_{\Delta\to 0}\lim_{t\to\infty}\Delta^{\delta-1/2}\,t\,[P_1(t)-2P_2(t)]=0$$

$\forall\,\delta>0$





Prob-ty of not escaping till time t.

$$\psi_{m,n} = \frac{\pi}{2} - \frac{m}{n}\pi, \quad m < n, \ (m,n) = 1, \ n < \left[\frac{2\pi}{\Delta}\right].$$

Clearly $N_t \subset M \setminus \bigcup_{k=0}^{n-1} T^{-k}(\hat{H}_1 \cup \hat{H}_2)$

if $t > 2\left[\frac{2\pi}{\Delta}\right]$

In view of Lemmas: $\underline{\psi = \psi_{m,n} + \eta}$, $|\eta| < \Delta/2$.

$\boxed{N_t}$

re of the set of orbits not escaping in time t

$\beta, \psi')$ and $x'' = (\beta, \psi'')$ be points of
er escaping periodic orbits with
) n' and n'' respectively. Then
x'' belong to different connected
nents of the set N_t if

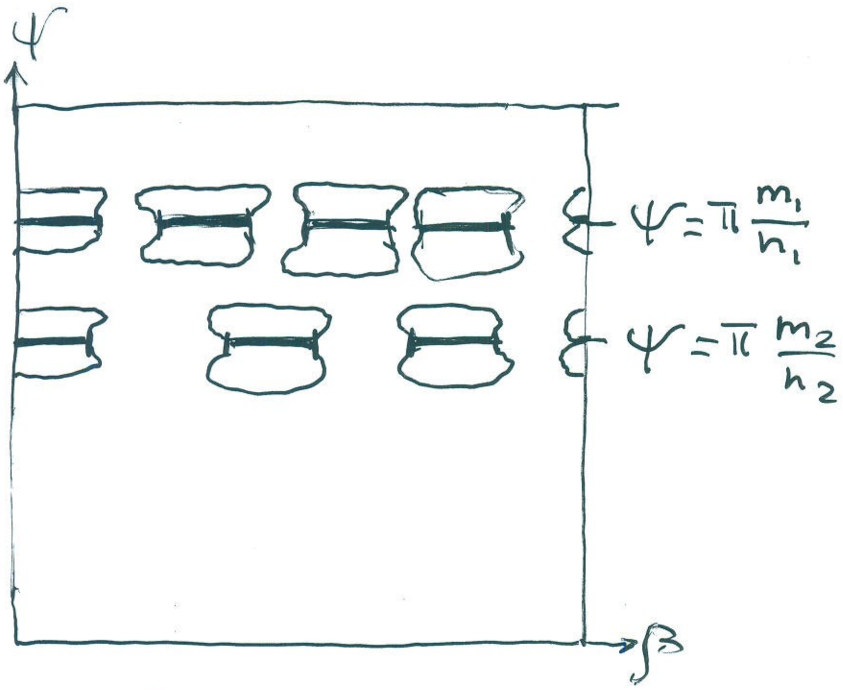
$(n' + n'' - 1) \max(\cos\psi', \cos\psi'')$

onempty connected components
$\forall t > 0$ contain a segment
$\psi) : \beta_1 \leq \beta \leq \beta_2,\ \psi = \psi_0\}$ consisting
er escaping periodic orbits.

fficiently large t N_t can

composed into the union of

[illegible]



- connected components of the set N_t of orbits which do not escape till time t.

- points which never escape

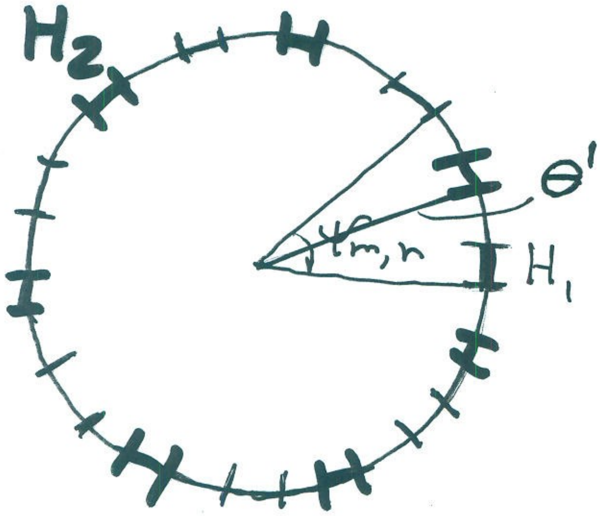
$$N_t = \bigcup_{\substack{m<n \\ (m,n)=1 \\ n<[2\pi/\Delta]}} \bigcup_{j} N^t_{\psi_{m,n},j}$$

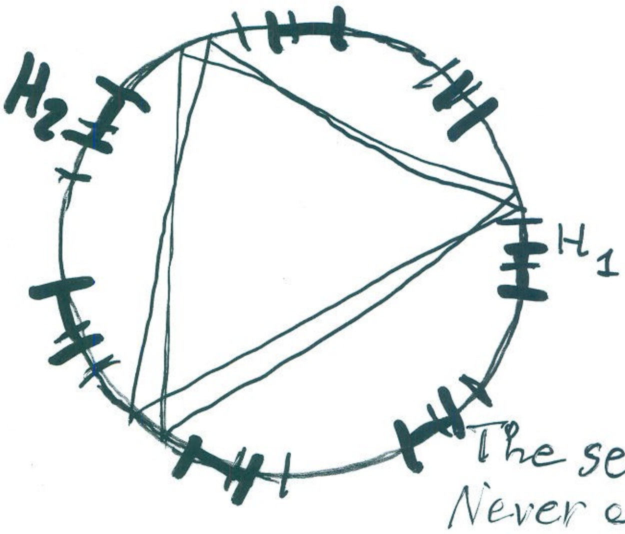
$$\mu(N_t) = \sum_{\substack{m<n \\ (m,n)=1 \\ n<[2\pi/\Delta]}} \sum_{k=0}^{n-1} \Bigg(\int_0^{\frac{2\pi}{n}-\theta'-\Delta} d\beta \int_{-\eta^-_{m,n}(\beta,k)}^{\eta^+_{m,n}(\beta,k)} \sin(\psi_{m,n}+\eta)\, d\eta \; +$$

$$+ \int_0^{\theta'-\Delta} d\beta \int_{-\eta^-_{m,n}(\beta,k)}^{\eta^+_{m,n}(\beta,k)} \sin(\psi_{m,n}+\eta)\, d\eta \quad .$$

Orbit of $(\beta, \psi_{m,n}+\eta)$ escapes not later than at time t if $\exists$ integer r, $0 \le r \le \left[\dfrac{t}{2\sin(\frac{m}{n}\pi-\eta)}\right]$ that $\operatorname{proj} T^r(\beta,\psi) \in H_1 \cup H_2$

$$\mu(N_t) \sim \frac{1}{4\pi} \sum_{\substack{m<n \\ (m,n)=1 \\ n<[2\pi/\Delta]}} \frac{n\left[g\left(\frac{2\pi}{n}-\theta'-\Delta\right)+g(\theta'-\Delta)\right] \sin^2 \frac{m}{n}\pi}{t} \qquad g(x) = \begin{cases} x^2, & x>0 \\ 0 & \text{otherwise} \end{cases}$$

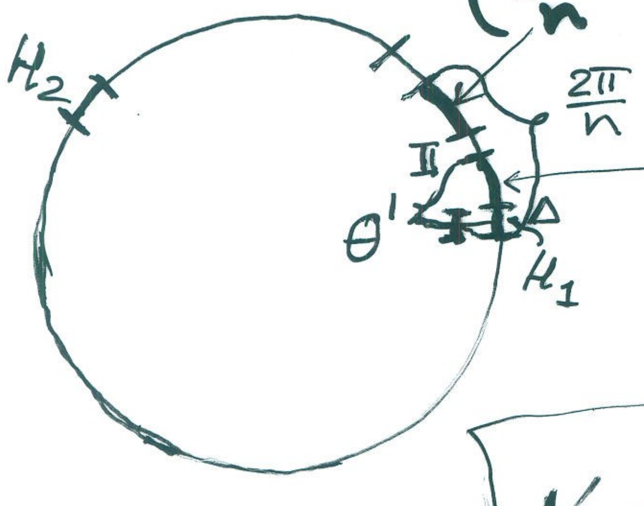




The set of Never escaping orbits consist of continuous families of periodic orbits

$$\theta' = \theta \left(\operatorname{mod} \frac{2\pi}{n}\right)$$

$$\left(\frac{2\pi}{n} - \theta' - \Delta\right)$$



$$(\theta' - \Delta)$$

Farey Numbers

$$\boxed{\mathcal{N}_t = \bigcup_{\substack{m<n \\ (m,n)=1 \\ n < \left[\frac{2\pi}{\Delta}\right]}} \bigcup_j \mathcal{N}^t_{\psi_{m,n},j}}$$

Th

$$P_\infty(\theta,\Delta) = \lim_{t\to\infty} t\,\mu(N_t) =$$

$$= \frac{1}{8\pi} \sum_{n=1}^{[2\pi/\Delta]} n\,(\phi(n) - \mu(n)) \left[g\left(\frac{2\pi}{n} - \theta' - \Delta\right) + g(\theta' - \Delta) \right]$$

$\phi(n)$:= Euler f-n
# of integers $0 < m \le n$ with $\gcd(m,n) = 1$

$\mu(n)$:= Möbius f-n
$\mu(1) = 1$, $\mu(p) = -1$ for primes p

Limit of small holes $\quad \Delta \to 0$

$$P_\infty(\theta, \Delta) = \lim_{t \to \infty} t\, \mu(N_t)$$

Mellin transform

$$\tilde{P}_\theta(s) = \int_0^\infty P_\infty(\theta, \Delta)\, \Delta^{s-1}\, d\Delta$$

exists if $\int_0^\infty |P_\infty(\theta, \Delta)|\, \Delta^{k-1}\, d\Delta$ for some $k > 0$

$P_\infty(\theta, \Delta) = 0$ if $\Delta > \pi$

Let $a>0$ is an integer

Consider $$\sum_{n \equiv a \pmod{q}} \frac{\phi(n)-\mu(n)}{n^{s+1}}$$

divide by $b=(a,q)$ – gcd

$$\boxed{\sum_{n'=a' \pmod{q'}} \frac{\phi(bn')-\mu(bn')}{(bn')^{s+1}}}$$

$n'=n/b$, $a'=a/b$, $q'=q/b$, where $(a';q')=1$

Dirichlet's characters $\chi(n)$ to modulus q are q-periodic multiplic-ve f-ns

Conjugacy classes mod q coprime to q form an abelian group (finite)

Order of this group equals $\phi(q)$

$\Rightarrow$ it has $\phi(q)$ irreducible represent-ns $\chi(n)$, $(n,q)=1$

Characters in this case are complex roots of unity $\chi(m)\chi(n)=\chi(mn)$

$\chi(n)=0$ if $(n,q)>1$.

Orthogonality relations

$$\frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(a)\chi(n) = \delta_{a,n}$$

$$\delta_{a,n} = \begin{cases} 1, \text{ if } a \equiv n \pmod q \\ 0, \text{ otherwise} \end{cases}$$

Inserting

$$\sum_{n \equiv a \pmod q} \frac{\phi(n) - \mu(n)}{n^{s+1}} = \frac{1}{\phi(q')} \sum_{\chi} \overline{\chi}(a') \sum_{n'=1}^{\infty} \chi(n') \frac{\phi(bn') - \mu(bn')}{(bn')^{s+1}}$$

Decompose into prime factors: $n' = \prod_p p^{\alpha_p}$

$$\Rightarrow \chi(n') = \prod_p \chi(p)^{\alpha_p}$$

Furthermore

$$\mu(bn') = \begin{cases} \mu(b) \prod_p (-1)^{\alpha_p}, \text{ if } bn' \text{ is square free} \\ 0, \text{ otherwise} \end{cases}$$

$$\phi(bn') = \phi(b) \prod_{p|n',\, p \nmid b} (1 - p^{-1})$$

$\alpha_p = 0$ if $p|b$

$$= \prod_p (1 - p^{-s-1}) = (\zeta(s+1))^{-1}$$

ius transform

$$\frac{\ldots n)}{\ldots +1} = (\zeta(s+1))^{-1} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(s+1)}$$

e

$$\frac{\ldots (n) - \mu(n))}{\ldots s+1} = \frac{\zeta(s) - 1}{\zeta(s+1)}$$

usly

$$\frac{\ldots (n)(\phi(n) - \mu(n))}{n^{s+1}} = \frac{L(s,\chi) - 1}{L(s+1,\chi)}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

et L-function

$$\sum \frac{\phi(n) - \mu(n)}{n^{s+1}} =$$

If $q' = 1 \Rightarrow L(s, \chi)$ reduces to $\zeta(s)$

$\forall q'$ $\exists$ trivial character: $\chi(a') = 1$
$\forall a', (a'; q') = 1$

$$\Rightarrow \quad L(s, 1) = \zeta(s) \prod_{p|q'} (1 - p^{-s})$$

$$\tilde{P}_{r/q}(s) = \frac{(2\pi)^{s+1}}{2s(s+1)(s+2)} \sum_{a=1}^{q} \frac{\left(1 - \left\{\frac{ar}{q}\right\}\right)^{s+2} + \left\{\frac{ar}{q}\right\}^{s+2}}{b^{s+1} \phi(q')} \times$$

$$\times \sum_{\chi} \frac{\bar{\chi}(a') (\phi(b) L(s, \chi) - \mu(b))}{L(s+1, \chi) \prod_{p|b} (1 - \chi(p) p^{-s-1})} \quad (*)$$

where $b = (a, q)$, $a' = a/b$, $q' = q/b$
(characters are taken mod q')

In $(*)$ odd characters $(\chi(-1) = -1)$ and their L-functions cancel

$\tilde{P}_{r/q}(s)$ has poles at $s = 0$, $s = -1$, $s = -2$, at zeros of $L(s+1, \chi)$ and at poles of $L(s, \chi)$

$L(s+1, \chi)$ with even χ has trivial zeros at
$s = -(2m+1)$, $m = 1, 2, \ldots$

All other (nontrivial) zeros of $L(s+1, \chi)$ have $\mathrm{Re}\, s = -1/2$ assuming extended RH

$$P_\infty(\theta,\Delta)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{ds\,\Delta^{-s}(2\pi)^{s+1}}{2s(s+1)(s+2)}\sum_{n=1}^{\infty}\frac{\phi(n)-\mu(n)}{n^{s+1}}\times$$

$$\| \quad \lim_{t\to\infty} t\mu(N_t)$$

$$\times\left[\left(1-\left\{\frac{n\theta}{2\pi}\right\}\right)^{s+2}+\left\{\frac{n\theta}{2\pi}\right\}^{s+2}\right]$$

Rational angles between holes

$\theta=2\pi\frac{r}{q}$, $(r,q)=1$

Single hole when $r=0$, $q=1$.

q	$\tilde{P}(s)$
1	$\dfrac{(2\pi)^{s+1}(\zeta(s)-1)}{2s(s+1)(s+2)\zeta(s+1)}$
2	$\dfrac{\pi^{s+1}\zeta(s)}{s(s+1)(s+2)\zeta(s+1)}$
3	$\dfrac{(2\pi/3)^{s+1}(3^s(7\zeta(s)+2^{s+2}(\zeta(s)-1)+2)-\zeta(s)(2^{s+2}+1)}{2s(s+1)(s+2)(3^{s+1}-1)\zeta(s+1)}$
4	$\dfrac{(\pi/2)^{s+1}(2^s(13\zeta(s)+3^{s+2}(\zeta(s)-1)+3)-\zeta(s)(3^{s+2}+5))}{4s(s+1)(s+2)(2^{s+1}-1)\zeta(s+1)}$
6	$[(\pi/3)^{s+1}(6^s+8.12^s-25.30^s+(1-3.2^s-13.3^s-8.4^s$ $+25.5^s+27.6^s-25.10^s+8.12^s-25.15^s+25.30^s)\zeta(s))]$ $\times[2s(s+1)(s+2)(2^{s+1}-1)(3^{s+1}-1)\zeta(s+1)]^{-1}$

Table 1. The function $\tilde{P}_{r/q}(s)$ (Eq. (30)) for $q = 1, 2, 3, 4, 6$ and $r = 1$.

q	s			
	1	-1	-2	-3
1	2	$-\frac{13}{12}$	$\frac{3}{2\pi}$	$\frac{119}{5760\pi^2\zeta'(-2)}$
2	1	$-\frac{1}{6}$	0	$-\frac{1}{720\pi^2\zeta'(-2)}$
3	1	$-\frac{1}{4}-\frac{5\ln 2}{9\ln 3}$	$\frac{3}{4\pi}$	$\frac{49}{5120\pi^2\zeta'(-2)}$
4	1	$-\frac{1}{3}-\frac{11\ln 3}{16\ln 2}$	$\frac{3}{\pi}$	$\frac{109}{1620\pi^2\zeta'(-2)}$
6	1	$[5\ln 5(10\ln 3-7\ln 5)+\ln 2(55\ln 5-76\ln 3)$ $+(10\ln 5-8\ln 2)(7\ln\Delta+12\zeta'(-1))]$ $\times[72\ln 2\ln 3]^{-1}$	$-\frac{3}{2\pi}$	$-\frac{79}{6400\pi^2\zeta'(-2)}$

Table 2. Some residues of $\tilde{P}_{r/q}(s)\Delta^{-s}$ given in table 1 divided by the factor Δ^{-s}. The $\ln\Delta$ appears for $q = 6$ due to a double pole at $s = -1$. There are also poles for further negative odd s, and along the critical line $\mathcal{R}e\ s = -1/2$. .

The simplest placements of two Roles

$\phi(q) \leq 2 \Rightarrow q = 1, 2, 3, 4$ and 6

Th. Consider a billiard in the unit circle with two holes $[0, \Delta]$ and $\left[2\pi \frac{r}{q}, 2\pi \frac{r}{q} + \Delta\right]$, where $q = 1, 2, 3, 4$ and 6, $0 < r < q$ are integers, $(r, q) = 1$.

If $t > f(t)\Delta^{-1}$, where $f(t) > 0$ and $\lim\limits_{t\to\infty} f(t) = \infty$ then

$$P_\infty\left(\frac{r}{q}, \Delta\right) = \lim_{t\to\infty} \mu(N_t) = \sum_k \operatorname*{Res}_{s=s_k} \tilde{P}_{r/q}(s)\Delta^{-s}.$$

$$\frac{\zeta(s)}{\zeta(s+1)} = \frac{-s}{2} \; \frac{\sin\frac{\pi s}{2}}{\sin\frac{\pi(s+1)}{2}} \; \frac{\zeta(1-s)}{\zeta(s)}$$

Assumethat RH holds

- each int-l $[\tau, \tau+1]$ contains t such that

$$|\zeta(\tfrac{1}{2}+it)| > \exp\left(-A\,\frac{\ln\tau \ln\ln\ln\tau}{\ln\ln\tau}\right)$$

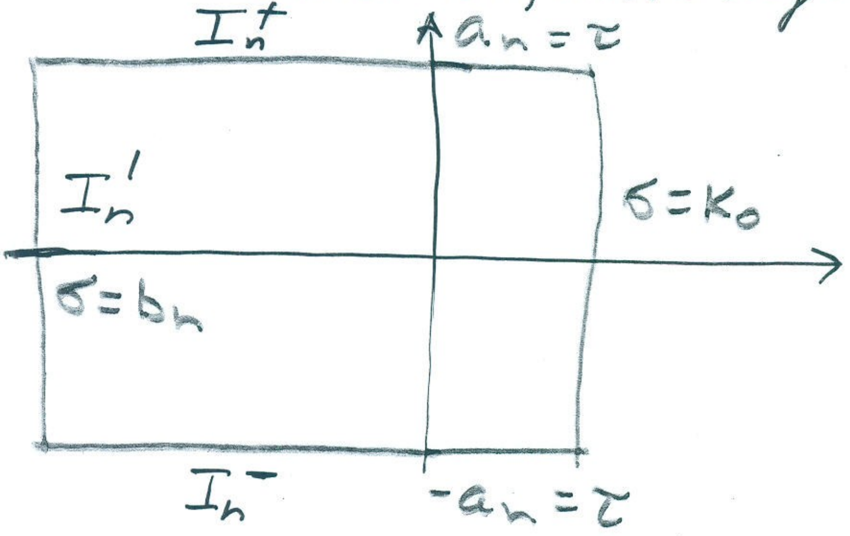
\# of zeros in $\{0<\sigma<1,\ 0<t<T\}$
$\zeta(\sigma+it)$

$$N(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} + O(T\ln T)$$

If a critical zero at $\frac{1}{2}+it$ has • $S(t)-1$

Consider infinite sequence C_n of contours

$s = \sigma + i\tau$



L. $\exists$ infinite sequence of contours C_n with $a_n \xrightarrow[n\to\infty]{} \infty$, $b_n \xrightarrow[n\to\infty]{} \infty$ such that

$$\lim_{n\to\infty} \int_{I_n' \cup I_n^+ \cup I_n^-} \widetilde{P}_{r/q}(s)\Delta^{-s}\,ds = 0 .$$

for any entry in the Table 1

(i.e. for $q = 1, 2, 3, 4, 6$)

L. Let $q = 1, 2, 3, 4$ or 6 then

$$\sum_j \operatorname{Res}_{s=s_j} \left(\widetilde{P}_{1/q}(s)\Delta^{-s}\right) < C\,\Delta\,|\ln \Delta|,$$

where $C > 0$ is a const, $s_j = -1, -2$ and all trivial zeros of $\zeta(s+1)$ (odd integers $m \le -3$)

L. Assume that RH is correct. Let $q = 1, 2, 3, 4$ or 6 then $\forall \alpha > 0$ $(r, q) = 1$.

$$C_1 \Delta^{1/2} < \sum_j \operatorname{Res}_{s=\frac{1}{2}+i\tau_j} \left(\widetilde{P}_{r/q}(s)\Delta^{-s}\right) < C_1 \Delta^{1/2 - s}$$

For irrational (w.r. to π) angles θ between the holes our analysys breaks down (poles on the critical line become dense, blocking analytic continuation?)

If θ is irrational $\Rightarrow$ fractional parts are uniformly distributed

assumption

Leading order behavior ("mean field" $\langle \cdot \rangle$)

$$\langle g\left(\frac{2\pi}{n}-\theta'-\Delta\right)+g\left(\theta'-\Delta\right)\rangle =$$

$$=\frac{n}{2\pi}\int_0^{2\pi/n}\left[g\left(\frac{2\pi}{n}-\theta'-\Delta\right)+g\left(\theta'-\Delta\right)\right]d\theta = \frac{n}{3\pi}\left(\frac{2\pi}{n}-\Delta\right)^3$$

$$\langle \phi(n)\rangle = 6n/\pi^2\,,\quad \langle \mu(n)\rangle = 0$$

$$tP(t)\approx\frac{1}{24\pi^2}\int_0^{2\pi/\Delta} n^2\,\frac{6n}{\pi^2}\left(\frac{2\pi}{n}-\Delta\right)^3 dn=\frac{1}{\Delta}$$

as required

Th. Consider an open circular billiard with one hole (i.e. with two holes of the same length Δ placed on top of each other). Let $P_1(t,\Delta)$ denotes prob-ty that particle will not escape till time t. Then

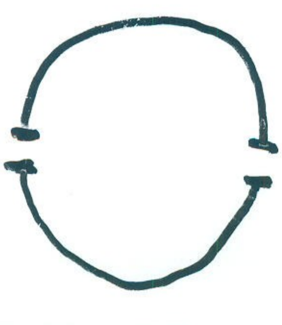
$$\forall \alpha > 0$$

$$RH \Leftrightarrow \lim_{\Delta \to 0} \lim_{t\to\infty} \Delta^{\alpha - 1/2} t \left[P_1(t,\Delta) - 2/\Delta \right] = 0.$$

Th. Consider an open circular Billiard with q holes of the same length Δ with the centers placed at the vertices of a right convex q-angle. Let $P_q(t,\Delta)$ is pr-ty that the particle will not escape through this system of q (different) holes and $P_1^{(q)}(t,\Delta)$:= pr-ty that particle will not escape till time t when all q holes are placed on top of each other. Then

$$\forall \alpha > 0$$

$$RH \Leftrightarrow \lim_{\Delta \to 0} \lim_{t\to\infty} \Delta^{\alpha - 1/2} t \left[P_1^{(q)}(t,\Delta) - q P_q(t,\Delta) \right] = 0$$



$q=2$

ture

ering all rational

$\theta = \frac{\pi}{2} - \pi \frac{m}{n}$ one gets

nt statements

extended RH,

Dirichlet's L-functions.

______ , ______________ .

e the term

zed RH

general L-functions
mberfields, elliptic curves...

eralized RH equivalent

Summary

By Drilling holes in the phase space of dynamical systems one can obtain (at least sometimes) a interesting and useful information about its dynamics .