ACCURACY ON EIGENVALUES FOR A SCHRODINGER OPERATOR WITH A DEGENERATE POTENTIAL

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1 Introduction

Let V be a nonnegative, real and continuous potential on $\mathbf{R}^d$, and h a small parameter.

The spectral asymptotics of the operator $H_h = -h^2\Delta + V$ on $L^2(\mathbf{R}^d)$ have been intensively studied.

Non degenerate case :

Assume $V(x) \to +\infty$ as $|x| \to +\infty$. Then H_h is essentially selfadjoint with compact resolvent , and the following semiclassical asymptotics hold, as $h \to 0$:

$$N(\lambda, H_h) \sim h^{-d}(2\pi)^{-d} v_d \int_{\mathbf{R}^d} (\lambda - V(x))_+^{d/2} dx \ . \tag{1}$$

$N(\lambda, H_h)$: number of eigenvalues less than a fixed energy λ. v_d : volume of the unit ball .

Remarks

1) The classical asymptotics are also given by the formula (1), provided we let $h = 1$ and $\lambda \to +\infty$.

2) In both cases :

asymptotic correspondance between :

the number of eigenstates with energy less than λ

and

the volume in phase space of the set
$S_\lambda = \{(x, \xi), f(x, \xi) \leq \lambda\}$,
where $f(x, \xi) = \xi^2 + V(x)$ is the principal symbol of H_h.

What about the degenerate case ?

If the potential V does not tend to infinity with $|x|$, the volume in phase space of S_λ may be infinite.

2 Min-max approach

A large class of degenerate potentials :

$X = (x, y) \in \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^d$, $d \geq 2$

$V(X) = f(x)g(y)$, $f \in C(\mathbf{R}^n; \mathbf{R}_+^*)$,
$g \in C(\mathbf{R}^m; \mathbf{R}_+)$,

(H1) for any $t > 0$ $g(ty) = t^a g(y)$ ($a > 0$ and $g(y) > 0$ for $y \neq 0$.

The spectrum of the operator $-\Delta_y + g(y)$ in $L^2(\mathbf{R}^m)$ is discrete and positive. Let us denote by μ_j its eigenvalues.

Remark 2.1 *If $f(x) \to +\infty$ as $|x| \to +\infty$* (H2), then $H_h = -h^2\Delta + V$ has a compact resolvent.

(H3) (local uniform regularity for f) :

$\exists\, b, c > 0$ s.t. $c^{-1} \leq f(x)$ and
$|f(x) - f(x')| \leq c f(x)|x - x'|^b$,

for any x, x' verifying $|x - x'| \leq 1$.

Theorem 2.2 *Let us assume the previous conditions on f and g. Then $N(\lambda; H_h)$ "behaves" like $h^{-n}(2\pi)^{-n}v_n n_{h,f}(\lambda)$*

where :

$n_{h,f}(\lambda) =$
$\int_{\mathbf{R}^n} \Sigma_{j \in \mathbf{N}}[\lambda - h^{2a/(2+a)} f^{2/(2+a)}(x)\mu_j]_+^{n/2} dx.$

Remark 2.3 *If moreover $f^{-m/a} \in L^1(\mathbf{R}^n)$ and $g \in C^1(\mathbf{R}^m \setminus \{0\})$, then the formula (1) holds.*

If there is some information on the growth of f, then the asymptotics can be computed in terms of power of h:

Remark 2.4 *If there exists $k > 0$ and $C > 0$ such that*

$\frac{1}{C}|x|^k \leq f(x) \leq C|x|^k$ *for* $|y| > 1$, *then*

$$\begin{array}{ll} if\, k > a & N(\lambda, H_h) \approx h^{-d} \\ if\, k = a & N(\lambda, H_h) \approx h^{-d} \ln \frac{1}{h} \\ if\, k < a & N(\lambda, H_h) \approx h^{-n-\frac{ma}{k}} \end{array}$$

There exists $\sigma, \tau \in]0,1[$ such that, for any $\lambda > 0$, one can find $h_0 \in]0,1[$, $C_1, C_2 > 0$ in order to have

$$(1 - h^\sigma C_1) N_{h,f}(\lambda - h^\tau C_2) \leq N(\lambda; H_h)$$

$$N(\lambda; H_h) \leq (1 + h^\sigma C_1) N_{h,f}(\lambda + h^\tau C_2) \quad \forall h \in]0, h_0[$$

$$N_{h,f}(\lambda) = h^{-n}(2\pi)^{-n} v_n n_{h,f}(\lambda)$$

$$n_{h,f}(\lambda) = \int_{\mathbf{R}^n} \Sigma_{j\in\mathbf{N}} [\lambda - h^{2a/(2+a)} f^{2/(2+a)}(y)\mu_j]_+^{n/2} dy.$$

If moreover one can find a constant C_3 such that, for any $\mu > 1$:

$$\int_{\{y, f(y) < 2\mu\}} f^{-p/a}(y) dy \leq C_3 \int_{\{y, f(y) < \mu\}} f^{-p/a}(y) dy$$

then take $C_2 = 0$ in the previous theorem :

$$(1 - h^\sigma C_1) n_{h,f}(\lambda) \leq N(\lambda; H_h) \leq (1 + h^\sigma C_1) n_{h,f}(\lambda)$$

$$\forall h \in]0, h_0[$$

3 Accurate estimates on eigenvalues

$$\widehat{H}_h = h^2 D_x^2 + h^2 D_y^2 + f(x)g(y) \qquad (2)$$

with $g \in C^\infty(\mathbf{R}^m \setminus \{0\})$ homogeneous of degree $a > 0$,

We replace assumptions (H2-H3) by :

$$\begin{aligned}
&f \in C^\infty(\mathbf{R}^n), \\
&\forall \alpha \in I\!N^n, \ (|f(x)|+1)^{-1}\partial_x^\alpha f(x) \in L^\infty(\mathbf{R}^n) \\
&0 < f(0) = \inf_{x\in\mathbf{R}^n} f(x) \\
&f(0) < \liminf_{|x|\to\infty} f(x) = f(\infty) \\
&\partial^2 f(0) > 0
\end{aligned} \qquad (3)$$

Homogeneity :

Define : $\hbar = h^{2/(2+a)}$.

Change y in $y\hbar$ and get :

$$sp\,(\widehat{H}_h) = \hbar^a \, sp\,(\widehat{H}^\hbar)\,, \qquad (4)$$

with $\widehat{H}^\hbar = \hbar^2 D_x^2 + D_y^2 + f(x)g(y)$.

$$\widehat{H}^\hbar = \hbar^2 D_x^2 + Q(x,y,D_y) \ :$$

$$Q(x,y,D_y) = D_y^2 + f(x)g(y) \,.$$

Denote the eigenvalues of $D_y^2 + g(y)$ by $(\mu_j)_{j>0}$.

By homogeneity the eigenvalues of $Q_x(y, D_y)$, for a fixed x, are given by the $(\lambda_j(x))_{j>0}$, where : $\lambda_j(x) = \mu_j \ f^{2/(2+a)}(x)$.
So we get :

$$\widehat{H}^\hbar \geq \left[\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)\right] . \qquad (5)$$

$$\inf \ sp_{ess}(\widehat{H}^\hbar) \geq \mu_1 f^{2/(2+a)}(\infty) \,. \qquad (6)$$

3.1 Born-Oppenheimer approximation :

"Effective " potential : $\lambda_1(x) = \mu_1\ f^{2/(2+a)}(x)$

Assumptions on $f \Longrightarrow$ existence of unique and nondegenerate well $U = \{0\}$,

with minimal value equal to μ_1.

Hence we can apply a theorem of A. Martinez and get :

Theorem 3.1 *For any $C > 0$, $\exists\ h_0 > 0$ s. t. for any $0 < \hbar < h_0$, the operator $(\widehat{H}^\hbar)$ admits a finite number of eigenvalues $E_k(\hbar)$ in $[\mu_1, \mu_1 + C\hbar]$, equal to the number of the eigenvalues e_k of $D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0)\, x,\ x >$ in $[0, +C]$ verifying :*

$$E_k(\hbar) = \lambda_k\left(\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)\right) + \mathbf{O}(\hbar^2)\,. \tag{7}$$

More precisely $E_k(\hbar) = \lambda_k(\widehat{H}^\hbar)$ has an asymptotic expansion

$$E_k(\hbar) \sim \mu_1 + \hbar\,(\,e_k + \sum_{j\geq 1} \alpha_{kj}\hbar^{j/2}\,). \tag{8}$$

If $E_k(\hbar)$ is asymptotically non degenerated, then there exists a quasimode

$$\phi_k^{\hbar}(x,y) \sim \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{j\geq 0} \hbar^{j/2} a_{kj}(x,y)\,, \quad (9)$$

Remarks

The previous formula implies

$$\lambda_k(\widehat{H}^{\hbar}) =$$
$$\mu_1 + \hbar\lambda_k\left(D_x^2 + \tfrac{\mu_1}{2+a} < \partial^2 f(0)\, x, x >\right) + \mathbf{O}(\hbar^{3/2})\,.$$

When $k = 1$, one can improve $\mathbf{O}(\hbar^{3/2})$ into $\mathbf{O}(\hbar^2)$.

The function ψ is defined by : $\psi(x) = d(x,0)$, where d denotes the Agmon distance related to the degenerate metric $\mu_1\ f^{2/(2+a)}(x)dx^2$.

3.2 Improving Born-Oppenheimer approximation :

Change of variables :

$$(x,\ y)\ \rightarrow\ (x,\ f^{1/(2+a)}(x)y)\ . \tag{10}$$

Change of test functions :

$u\ \rightarrow\ f^{-m/(4+2a)}(x)u\ ,$

$\Longrightarrow$ get a unitary transformation.

Thus :

$$sp\,(\widehat{H}^{\hbar})\ =\ sp\,(\widetilde{H}^{\hbar}) \tag{11}$$

where $\widetilde{H}^{\hbar}$ is the self-adjoint operator on $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ given by

$$\widetilde{H}^{\hbar}\ =\ \hbar^2 L^{\star}\ L\ +\ f^{2/(2+a)}(x)\left(D_y^2 + g(y)\right)\ , \tag{12}$$

with

$$L(x,y,D_x,D_y)\ =$$

$$D_x\ +\ \frac{1}{(2+a)f(x)}[(yD_y)\ -\ i\frac{m}{2}]\nabla f(x)\ .$$

Decompose $\widetilde{H}^\hbar$ in four parts :

$$
\begin{aligned}
&\widetilde{H}^\hbar \;=\; \hbar^2 D_x^2 \;+\; f^{2/(2+a)}(x)\left(D_y^2 + g(y)\right) \\
&+\hbar^2 \tfrac{2}{(2+a)f(x)} (\nabla f(x) D_x)(y D_y) \\
&+i\hbar^2 \tfrac{1}{(2+a)f^2(x)} \left(|\nabla f(x)|^2 - f(x)\Delta f(x)\right) [(y D_y) + i\tfrac{m}{2}] \\
&+\hbar^2 \tfrac{1}{(2+a)^2 f^2(x)} |\nabla f(x)|^2 [(y D_y)^2 + \tfrac{m^2}{4}]
\end{aligned} \tag{13}
$$

Our goal : prove that the only significant role up to order 2 in $\hbar$ is played by the first operator, namely :
$\widetilde{H}_1^\hbar \;=\; \hbar^2 D_x^2 \;+\; f^{2/(2+a)}(x)\left(D_y^2 + g(y)\right)$.

Denote by $\nu_{j,k}^\hbar$ the eigenvalues of the operator $\hbar^2 D_x^2 \;+\; \mu_j f^{2/(2+a)}(x)$ and by $\psi_{j,k}^\hbar$ the associated normalized eigenfunctions .

Consider the following test functions :

$$u_{j,k}^\hbar(x,y) = \psi_{j,k}^\hbar(x)\varphi_j(y) \;,$$

$$D_y^2\varphi_j(y) \;+\; g(y)\varphi_j(y) \;=\; \mu_j\varphi_j(y) \;.$$

We have immediately :

$$\widetilde{H}_1^\hbar(u_{j,k}^\hbar(x,y)) \;=\; \nu_{j,k}^\hbar u_{j,k}^\hbar(x,y) \;.$$

Theorem 3.2 *For any fixed integer* $N > 0$, *there exists a positive constant* $h_0(N)$ *verifying : for any* $\hbar \in]0, h_0(N)[$, *for any* $k \leq N$ *and any* $j \leq N$ *such that*

$$\mu_j < \mu_1 f^{2/(2+a)}(\infty) ,$$

there exists an eigenvalue $\lambda_{jk} \in sp_d\left(\widehat{H}^{\hbar}\right)$ *such that*

$$| \lambda_{jk} - \lambda_k \left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)\right) | \leq \hbar^2 C . \tag{14}$$

Consequently, when $k = 1$, *we have*

$$| \lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}}\right] | \leq \hbar^2 C . \tag{15}$$

Sketch of the proof .

Prove that :

$$\|(\widehat{H}^{\hbar} - \widetilde{H}_1^{\hbar})(u_{j,k}^{\hbar}(x,y))\| =$$

$$\|(\widehat{H}^{\hbar} - \nu_{j,k}^{\hbar})u_{j,k}^{\hbar}(x,y)\| = \mathbf{O}(\hbar^2) .$$

Lemma 3.3 . *For any integer* N , *there exists a positive constant* $C = C(N)$ *such that for any* $k \leq N$, *the eigenfunction* $\psi_{j,k}^{\hbar}$ *satisfies the following inequalities : for any* $\alpha \in I\!N^n$, $|\alpha| \leq 2$,

$$\begin{aligned} \| \hbar_j^{|\alpha|/2} |D_x^{\alpha} \psi_{j,k}^{\hbar}| \| &< C \\ \| \left(\frac{\nabla f(x)}{f(x)}\right)^{\alpha} \psi_{j,k}^{\hbar} \| &< \hbar_j^{|\alpha|/2} C \end{aligned} \tag{16}$$

with $\hbar_j = \hbar \mu_j^{-1/2}$.

3.3 Middle energies

Assume : $a \geq 2$ and $f(\infty) = \infty$, and $g \in C^\infty(\mathbf{R}^m)$.

Goal : Refine the preceding results and get sharp localization near the μ_j's for much higher values of j's.

Theorem 3.4 *If j is such that $\mu_j \leq \hbar^{-2}$, then for any integer N , there exists $C = C(N)$ depending only on N such that, for any $k \leq N$, there exists an eigenvalue $\lambda_{jk} \in sp_d\left(\widehat{H}^\hbar\right)$ such that*

$$| \lambda_{jk} - \lambda_k \left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)\right) | \leq C\mu_j \hbar^2 . \tag{17}$$

Consequently, when $k = 1$, we have

$$| \lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}}\right] | \leq C\mu_j \hbar^2 . \tag{18}$$

4 An application

We consider a Schrödinger operator on $L^2(\mathbf{R}^d_z)$ with $d \geq 2$,

$$\widehat{P}_h = -h^2\Delta + V(z)$$
$$V \in C^\infty(\mathbf{R}^d\,;\,[0,+\infty[)$$
$$\liminf_{|z|\to\infty} V(z) > 0$$
$$\Gamma = V^{-1}(\{0\}) \quad \text{is a regular hypersurface.} \tag{19}$$

More assumptions :

(H1) Γ is connected and
$\exists\, m \in I\!N^\star$ and $C_0 > 0$ s.t.

$$C_0^{-1} d^{2m}(z,\,\Gamma) \leq V(z) \leq C_0\, d^{2m}(z,\,\Gamma)$$
$$\forall\, z\,;\ d(z,\,\Gamma) < C_0^{-1}\,.$$

Choose an orientation on Γ and then a unit normal vector $N(s)$ on each $s \in \Gamma$.

Define the function on Γ :

$$f(z) = \frac{1}{(2m)!}\left(N(z)\frac{\partial}{\partial z}\right)^{2m} V(z)\,, \quad \forall\, z \in \Gamma\,. \tag{20}$$

(H2) f achieves its minimum on Γ on a finite number of discrete points:

$$\Sigma_0 = f^{-1}(\{\eta_0\}) = \{s_1, \ldots, s_{\ell_0}\},$$
$$\text{if } \eta_0 = \min_{s\in\Gamma} f(s).$$

(H3) The hessian of f at each point $s_j \in \Sigma_0$ is non degenerated.

Then, $f(s) > 0, \quad \forall s \in \Gamma$.

$Hess(f)_{s_j}$ has $d-1$ non negative eigenvalues

$$\rho_1^2(s_j) \leq \ldots \leq \rho_{d-1}^2(s_j), \qquad (\rho_j(s_j) > 0).$$

We denote by $(\mu_j)_{j\geq 1}$ the increasing sequence of the eigenvalues of the operator $-\dfrac{d^2}{dt^2} + t^{2m}$ on $L^2(\mathbf{R})$,

and by $(\varphi_j(t))_{j\geq 1}$ the associated orthonormal Hilbert base of eigenfunctions.

Theorem 4.1 *For any* $N \in \mathbb{N}^*$,
there exist $\hbar_0 \in [0,1]$ *and* $C_0 > 0$ *such that,*
if $\mu_j << \hbar^{-2m/(2m^2+3m+1)}$,
and if $\alpha \in \mathbb{N}^{d-1}$ *and* $|\alpha| \leq N$,
then $\forall\, s_\ell \in \Sigma_0$, $\exists\, \lambda_{j\,\ell}^{\hbar} \in sp_d(\widehat{P}_h)$ *s.t.*

$$\left|\, \lambda_{j\,\ell}^{\hbar} - \hbar^{2m/(m+1)} \left[\eta_0^{1/(m+1)} \mu_j + \hbar^{1/(m+1)} \mu_j^{1/2} (\, \mathcal{A}_\ell) \right] \right.$$

$$\leq \hbar^2 \mu_j^{(4m+3)/2m} C_0 \,.$$

$\mathcal{A}_\ell =$
$\frac{1}{\eta_0^{m/(2m+2)}(m+1)^{1/2}} \left[2\alpha\rho(s_\ell) + Tr(Hess(f(s_\ell)))\right]$
$(\alpha\rho(s_\ell) = \alpha_1\rho_1(s_\ell) + \ldots \alpha_{d-1}\rho_{d-1}(s_\ell)\,)$.