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ON THE SPECTRUM IN SMILANSKY'S MODEL OF IRREVERSIBLE QUANTUM GRAPHS

- Smilansky's model
- One-oscillator case: 1. Self-adjointness

  2. Small α

  3. Large α
- References

SMILANSKY'S MODEL

Interaction between a quantum graph $\{\Gamma; \Delta\}$ and a system of $K \geq 1$ harmonic oscillators (describing "environment")

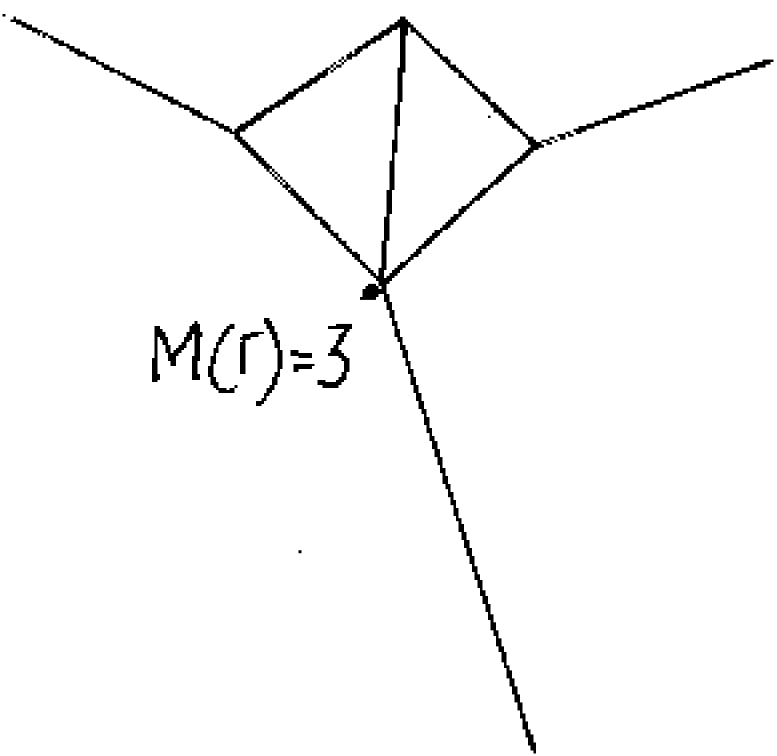
CLASS OF METRIC GRAPHS: $\Gamma \in \mathcal{G}$ **if**

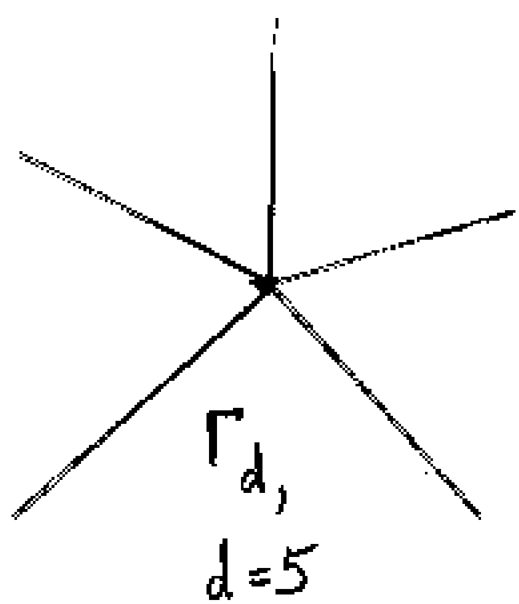
Γ is connected; Γ is a compact graph, plus may be a finite number of infinite leads (each isometric to the half-line).

$M = M(\Gamma)$ notation for the number of leads; $M \geq 0$.

MODEL CASE: $\Gamma = \Gamma_d$ – star graph with $d \geq 1$ infinite edges.

$$M(\Gamma_d) = d.$$





OPERATOR $\mathbf{A}_\alpha$

The space: $H = L^2(\Gamma \times \mathbb{R}^K)$. **Notation:** $x \in \Gamma$,

$q = (q_1, \dots, q_K) \in \mathbb{R}^K$ – **generic point;**

$o_1, \dots, o_K \in \Gamma$ – **pts chosen ('k-th oscillator is attached at o_k')**

The operator: on each edge

$$\mathbf{A}_\alpha U(x,q) = -U''_{xx} + \sum_{k=1}^{K} \frac{\nu_k^2}{2}(-U''_{q_k q_k} + q_k^2 U);$$

Kirchhoff conditions $[U'_x](v) = 0$ **at the vertices** $v \neq o_k$;

Dirichlet, or Neumann condition at $\partial\Gamma$;

Conditions at $x = o_k$ **(they describe the interaction):**

$$[U'_x](o_k, q) = \alpha_k q_k U(o_k, q), \qquad q \in \mathbb{R}^K.$$

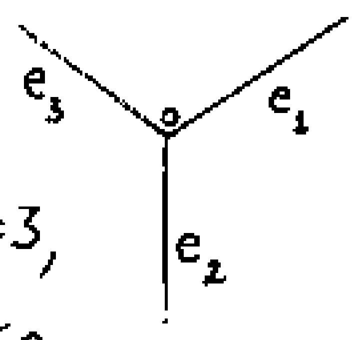
$\alpha = (\alpha_1, \dots, \alpha_K)$ – **the coupling parameters,** $\alpha_k \geq 0$.

The value of α_k **expresses the strength of interaction between the graph and the** k**-th oscillator.**

α **is involved only in the conditions at** o_k;

the action of $\mathbf{A}_\alpha$ **is the same for all** α.

$$[f'](o) = \sum_{j=1}^{d(o)} f'_k(o)$$



$d(o) = 3$,

$f_k = f \restriction e_k$

PROBLEM:

TO DESCRIBE THE SPECTRUM OF $\mathbf{A}_\alpha$ FOR ALL α

The leading case: $K = 1$.

$$\mathbf{A}_{\alpha,\nu} U(x,q) = -U''_{xx} + \frac{\nu^2}{2}(-U''_{q^2} + q^2 U);$$

$$[U'_x](o,q) = \alpha q U(o,q), \qquad q \in \mathbb{R}.$$

Scaling reduces the general case to $\nu = 1$.

Main parameters: $d = \deg(o) \geq 1$; $M = M(\Gamma) \geq 0$.

BELOW THE RESULTS ARE FORMULATED FOR THE GENERAL CASE, BUT CALCULATIONS (IF ANY) ARE GIVEN FOR $\nu = 1$.

THE OPERATOR $\mathbf{A}_0$: separation of variables.

Everything can be described in the explicit terms.

For $\alpha \neq 0$ it is natural to try perturbation theory.

HOWEVER, THE PERTURBATION IS TOO STRONG!!!

In terms of quadratic forms it is only form-bounded but not form-compact.

The standard approaches do not work.

REDUCTION TO AN INFINITE SYSTEM OF ODE

$\chi_n(q)$ – **Hermite functions, normalized in $L^2(\mathbb{R})$.**

$$U(x,q) = \sum_{n\in\mathbb{N}_0} u_n(x)\chi_n(q) \qquad (U \sim \{u_n\}).$$

Then $\mathbf{A}_\alpha U \sim \{(-u_n'' + (1/2+n)u_n\}$ on each edge,

with the prescribed boundary cond. at vertices $v \neq o$.

Matching condition at o turns into

$$[u_n'](o) = \frac{\alpha}{\sqrt{2}}\left(\sqrt{n+1}\,u_{n+1}(o) + \sqrt{n}\,u_{n-1}(o)\right). \qquad (1$$

For $\alpha = 0$ this is Kirchhoff cond., and for GENERAL ν

$$\mathbf{A}_0 = \sum_{n\in\mathbb{N}_0}{}^{\oplus}\left(-\Delta_\Gamma + \nu^2(1/2+n)\right)$$

where Δ_Γ is the Laplacian on Γ.

HENCE: If $M = 0$ (*compact* Γ), then $\sigma(\mathbf{A}_0)$ is DISCRETE

If $M > 0$ (*non-compact* Γ), then $\sigma_{a.c.}(\mathbf{A}_0) = [\nu^2/2, \infty)$;

$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0) = Mn \qquad \text{for } |\lambda - n\nu^2| < \nu^2/2,\ n \in \mathbb{N}.$$

Also, $\sigma(\mathbf{A}_0) = [\nu^2/2, \infty)$; embedded eigenvalues are possible.

SELF-ADJOINT REALIZATION OF $\mathbf{A}_\alpha$

THEOREM 1. *THE OPERATOR* $\mathbf{A}_\alpha$ *IS SELF-ADJOINT ON THE DOMAIN* $\mathcal{D}_\alpha$ *DEFINED AS FOLLOWS:*

$U \sim \{u_n\} \in \mathcal{D}_\alpha$ *IFF*

1. $u_n \in H^2(e)$ *ON EACH EDGE OF* Γ*;*

2. u_n *IS CONTINUOUS ON* Γ*; BDRY COND. ON* $\partial\Gamma$ *AND THE CONDITIONS (1) ARE SATISFIED FOR ALL* n*;*

3.
$$\sum_{n\in\mathbb{N}_0}\int_\Gamma \left|-u_n''+\nu^2(n+1/2)u_n\right|^2 dx < \infty.$$

IDEA OF PROOF: v.Neumann procedure (calculation of the deficiency indices)

We have to show that $U = 0$ is the only L^2-solution of the equation $\mathbf{A}_\alpha U = \Lambda U$, $\Lambda \notin \mathbb{R}$. This reduces to

$$-u_n'' + (n + 1/2 - \Lambda)u_n = 0$$

on each edge.

If $\Gamma = \Gamma_d$ (STAR GRAPH WITH ALL ITS d EDGES OF INFINITE LENGTH), then on each edge

$$u_n(t) = C_n e^{-t\sqrt{n+1/2-\Lambda}}.$$

Due to the condition at $x = o$, this leads to

$$\sqrt{n+1}C_{n+1} + d\alpha^{-1}\sqrt{2n+1-2\Lambda}\,C_n + \sqrt{n}C_{n-1} = 0.$$

This is a recurrence system, with a Jacobi matrix. For any $\Gamma \in \mathcal{G}$, we come to a similar system, with d replaced by $M(\Gamma)$ and an EXPONENTIALLY SMALL correction in the coefficient

The system is analyzed with the help of Birkhoff – Adams theorem which gives the desired result

SPECTRUM OF $\mathbf{A}_\alpha$ for $\alpha > 0$

For $\alpha\sqrt{2} < d\nu$ (small α) and for $\alpha\sqrt{2} > d\nu$ (large α)

THE RESULTS ARE DIFFERENT

(in the talk I skip the borderline case $\alpha\sqrt{2} = d\nu$)

I. SMALL α: VARIATIONAL APPROACH (for $\nu = 1$)

Quadratic form of $\mathbf{A}_\alpha$: $\mathbf{a}_\alpha[U] = \mathbf{a}_0[U] - \alpha \mathbf{b}[U]$;

$$\mathbf{a}_0[U] = \int_\Gamma (|u_n'|^2 + (n+1/2)|u_n|^2)dx,$$

$$\mathbf{b}[U] = \sum_{n\in\mathbb{N}} \sqrt{2n}\,\mathrm{Re}(u_n(o)\overline{u_{n-1}(o)}).$$

$\mathbf{a}_0[U]$ is positive definite and closed on its natural domain $\mathfrak{d}$. The corresponding s.a. operator is $\mathbf{A}_0$.

LEMMA.

$$d|\mathbf{b}_\alpha[U]| \leq \sqrt{2}\mathbf{a}_0[U] + C\|U\|^2_{L^2}, \qquad \forall\, U \in \mathfrak{d},$$

WITH C DEPENDING ON THE STRUCTURE OF Γ

AND ON THE BDRY COND. ON ∂Γ.

IN PARTICULAR, $C = 0$ FOR $\Gamma = \Gamma_d$.

THE CONSTANT $\sqrt{2}$ IS SHARP.

THEREFORE, FOR α SMALL THE Q. FORM

$$\mathbf{a}_\alpha[U]$$

IS BOUNDED BELOW AND CLOSED ON $\mathfrak{d}$.

THE CORRESP. S.A. OPERATOR IS $\mathbf{A}_\alpha$.

For α large, the operator $\mathbf{A}_\alpha$ is unbounded below

THEOREM 2. *LET $\alpha\sqrt{2} < d\nu$. THEN*

1. $\sigma_{a.c.}(\mathbf{A}_\alpha) = \sigma_{a.c.}(\mathbf{A}_0)$, $\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha) = \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0)$.

MOREOVER, FOR THE PAIR $(\mathbf{A}_\alpha, \mathbf{A}_0)$ *THERE EXIST THE COMPLETE ISOMETRIC WAVE OPERATORS, AND THE SAME IS TRUE FOR THE PAIR* $(\mathbf{A}_0, \mathbf{A}_\alpha)$.

2. SPECTRUM of $\mathbf{A}_\alpha$ *BELOW* $\nu^2/2$ *IS FINITE AND*

$$N_-(\nu^2/2; \mathbf{A}_\alpha) \sim C(d\nu - \alpha\sqrt{2})^{-1/2}, \qquad (2)$$

WITH AN EXPLICITLY GIVEN CONSTANT C.

IDEA OF PROOF OF (2):

reduction to the eigenvalue asymptotics for a certain zero-diagonal Jacobi matrix J with the non-diagonal entries

$$j_{n,n+1} = j_{n+1,n} = 1/2 + b_n, \qquad b_n \to 0,$$

$b_n = b_n(d)$+exp. *small error depending on* Γ.

The spectrum of J consists of $[-1, 1]$ (a.c. part) and of eigenvalues $\pm\lambda_n$, $\lambda_n \searrow 1$.

It turns out that $N_-(\nu^2/2; \mathbf{A}_\alpha) = N_+(\frac{d\nu}{\alpha\sqrt{2}}; \mathbf{J})$ (+1).

II. LARGE α (for $\Gamma = \mathbb{R}$ in cooperation with S.N. Naboko)

THEOREM 3. *LET* $\alpha\sqrt{2} > d\nu$. *THEN* $\sigma_p(\mathbf{A}_\alpha) \cap (-\infty, \nu^2/2) = \emptyset$;

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = \mathbb{R}, \qquad \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha) = \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0) + 1$$

PHASE TRANSITION: σ_p below $\nu^2/2$ disappears.

An additional branch of $\sigma_{a.c.}$ appears instead.

This effect was interpreted by Smilansky as

IRREVERSIBILITY OF THE SYSTEM.

IDEA OF PROOF: analysis of $(\mathbf{A}_\alpha - \Lambda)^{-1}$ for $\Lambda \notin \mathbb{R}$.

We use the connection between the a.c. spectrum of a s.a. operator and the jump of its (bordered) resolvent when Λ crosses the real line.

KEY FORMULA

$$(\mathbf{A}_\alpha - \lambda)^{-1} - (\mathbf{A}_0 - \lambda)^{-1} = \mathbf{T}(\lambda)\Big(\alpha\mathbf{J}(\lambda;\alpha)^{-1} - (d\mathbf{P}(\lambda))^{-1}\Big)\mathbf{T}(\overline{\lambda})^*$$

Here $\mathbf{T}(\lambda) : \ell^2 \to L^2(\Gamma \times \mathbb{R})$ – a nice operator-valued function, analytic in the upper and in the lower half-planes.

$\mathbf{J}(\lambda;\alpha)$ – a Jacobi analytic matrix-valued function

$\mathbf{P}(\lambda)$ – an analytic diagonal matrix-valued function.

Key formula reduces study of the jump of $(\mathbf{A}_\alpha - \lambda)^{-1}$ to the analysis of the jump of $\mathbf{J}(\lambda;\alpha)^{-1}$.

THE ENTRIES OF $\mathbf{J}(\lambda;\alpha)$ AND OF $\mathbf{P}(\lambda)$ FOR AN ARBITRARY $\Gamma \in \mathcal{G}$ DIFFER FROM THOSE FOR $\Gamma = \Gamma_0$ BY EXPONENTIALLY SMALL TERMS

For α small the jump of $\mathbf{J}(\lambda;\alpha)^{-1}$ is 0 a.e.,

For α large it is a rank one operator a.e. –

this leads to the result of Theorem 3.

Analysis of the boundary behaviour of $\mathbf{J}(\Lambda;\alpha)$ is the most difficult part of the work. Subtle results of the theory of operator-valued analytic functions are used, including some older Naboko's results.

CONCLUSIONS (for $K = 1$)

1. For $\mathbf{A}_0$ the point spectrum is unstable. The structure of $\sigma_{a.c.}(\mathbf{A}_0)$ is determined ONLY by the numbers ν and $M = M(\Gamma)$.

2. The point α^* of the phase transition is determined ONLY by the numbers ν and $d = \deg(o)$:

$$\alpha^*\sqrt{2} = d\nu.$$

3. The behaviour of $N_-(\nu^2/2; \mathbf{A}_\alpha)$ as $\alpha \nearrow \alpha^*$ is determined ONLY by ν, d and does not depend on the edge lengths.

LOCALITY PRINCIPLE:

only the numbers ν, $M(\Gamma)$ and $d = \deg(o)$ are responsible for the stable characteristics of $\sigma(\mathbf{A}_\alpha)$.

LIST OF REFERENCES

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