On possible spectral structure of linear continuous operators

Gallia est omnis divisa in partes tres

Ceasar, Bellum Gallicum

X – a topological vector space (over $\mathbb{C}$)

$T : X \to X$ – a continuous linear operator

$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not (continuously) invertible}\}$;

$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = (\lambda I - T)^{-1}(0) \neq \{0\}\}$;

$\sigma_c(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : (\lambda I - T)(X) \text{ is dense in } X\}$;

$\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$;

$\sigma_p^n(T) = \{\lambda \in \mathbb{C} : \dim \ker(\lambda I - T) \geq n\}$;

$\sigma_{p,n}(T) = \{\lambda \in \mathbb{C} : \dim \ker(\lambda I - T) = n\}$.

Question 1. For a given class $\mathcal{X}$ of topological vector spaces, which triples of subsets of $\mathbb{C}$ are point, continuous and residual spectra of continuous linear operators acting on a space from $\mathcal{X}$.

1. Necessary conditions

Theorem 1. Let X be a separable Fréchet space and $T: X \to X$ be a ~~continuous~~ closed densely defined linear operator. Then the sets

$\sigma_p^n(T)$ are Souslin $n=1,\dots,\infty$,

$\sigma_c(T)$, $\sigma_r(T)$ and $\underbrace{\sigma_{p,n}(T)}_{(n<\infty)}$ are co-Souslin

Moreover $\exists$ an F_σ-set D : $\sigma_p(T) \cup \sigma_r(T) = \sigma_p(T) \cup D$

Remark. If $\sigma_{p,\infty}(T)$ is a Borel measurable set, then all above sets are Borel measurable

Theorem 2. Let X be a reflexive separable Banach space and T be a closed densely defined linear operator acting on X. Then the set

$\sigma_c(T)$ is G_δ

$\sigma_p^n(T)$ is F_σ for any $n=1,2,\dots$.

Remark. Reflexivity can be replaced by quasireflexivity.

2. „Spectral syntesis"

Theorem 3 Let $A_1, A_2, \dots$ be a decreasing sequence of F_σ-sets (subsets of $\mathbb{C}$), A_0 be a G_δ-set, and $A_{-1} \subset \mathbb{C}$ be such that

A_{-1}, A_0, A_1 are disjoint

$A_{-1} \cup A_0 \cup A_1$ is a non-empty compact set. Then there exists a continuous linear operator T on ℓ_2 such that $\sigma_r(T) = A_{-1}$, $\sigma_c(T) = A_0$ and $\sigma_p^n(T) = A_n$, $n=1,2,\dots$

Theorem 4. Let $K \subset \mathbb{C}$ be a non-empty compact set, being a disjoint union of A, B and C, where A is Souslin, B is co-Souslin and there exists an F_σ-set D for which $A \cup D = A \cup C$. Then there exists a separable Banach space X and a continuous linear operator $T: X \to X$ for which $\sigma_p(T) = A$, $\sigma_c(T) = B$ and $\sigma_r(T) = C$.

Theorem 5 Let $A_1, A_2, \ldots$ be a decreasing sequence of Borel sets, A_0, A_{-1} be Borel sets such that A_{-1}, A_0, A_1 are disjoint $A_{-1} \cup A_0 \cup A_1$ is a non-empty compact set and $\exists$ an F_σ-set D for which $A_1 \cup A_{-1} = A_1 \cup D$. Then there exists a separable Banach space X and a continuous linear operator $T: X \to X$ for which $\sigma_r(T) = A_{-1}$, $\sigma_c(T) = A_0$ and $\sigma_p^n(T) = A_n$, $n = 1, 2, \ldots$

Theorem 6 Let X be a separable Fréchet space and T be a closed densely defined linear operator acting on X. Then $\sigma(T)$ is a $G_{\delta,c}$-set. Conversely for any G_{δ}-set $A \subseteq \mathbb{C}$ there exists a separable Fréchet space X and a continuous linear operator $T: X \to X$ for which

$$\sigma(T) = A.$$

of subsets of $\mathbb{C}$ there exists T acting on $X \in \mathcal{X}$ for which

$$\sigma_r(T) = A_{-1}, \quad \sigma_c(T) = A_0, \quad \sigma_p^n(T) = A_n, \; n \geq 1$$

History:

1) G. Kalisch [1972]: for any nonempty compact set $K \subseteq \mathbb{R}$ there exists a LCO T on a separable Hilbert space for which

$$\sigma(T) = \sigma_p(T) = \sigma_{p,1}(T) = K$$

2) L. Nikol'skaia [1974]: the point spectrum of a closed densely defined linear operator acting on a separable reflexive Banach space is an F_σ-set. Any F_σ-set is the point spectrum of such an operator acting on a separable Hilbert space

3) R. Kaufman [1981, 1985]: the point spectrum of a continuous linear operator acting on a separable Banach space is a Souslin set. Any bounded Souslin set is the point spectrum of such an operator.

O. Smolyanov and S. Shkarin 1999

S. Shkarin 2001