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Inverse Boundary Spectral Problem for a Riemannian Polyhedron

with

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Outline

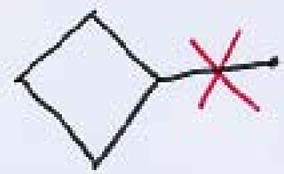
- Basic definitions. Admissible Riemannian Polyhedron.
- Formulation of the Problem.
- Gaussian beams on interface.
- Sketch of the Uniqueness proof.

Examples: We consider multi-component body, each component consists of different material with different characteristics (geometric properties), say, in geophysics, clay, rock, oil, in medicine, human body consists of fat, skin, muscles, bones.

Admissible Polyhedron X

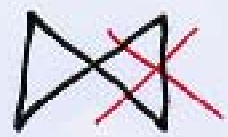
dimensionally homogeneous $dim = n$, i.e. each $(n-k)$-simplex is contained in at least one (n)-simplex, $k = 1, 2, ..., n$.

(n)-simplices are called **CHAMBERS**, $(n-1)$-simplices are called **FACES**



closed, connected, finite

$(n-1)$-chainable i.e. any two chambers can be joined by a chain of faces and chambers

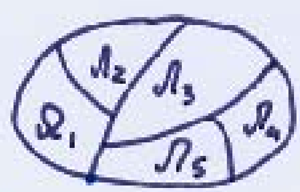


FACES

interface boundary γ adjacent to two chambers

part of the polyhedron boundary Γ adjacent to one chamber

CHAMBERS



CONICAL POINTS

Admissible Riemannian Polyhedron $\mathcal{M}$

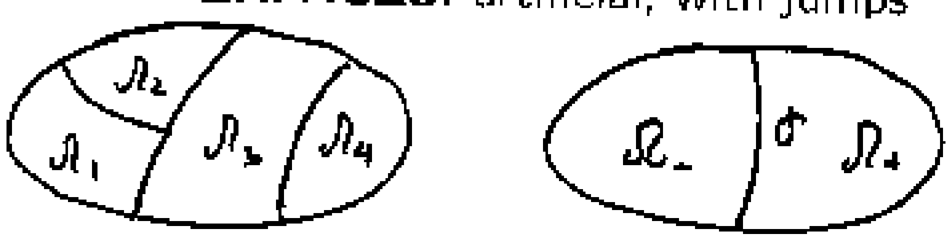
We endow AP X with a family of smooth up to the boundary (non-degenerate, Riemannian) **metrics**, not necessary glued by isometry:

$$X \to \{g_m\}_1^M \to \mathcal{M}$$

$$g_i|_\gamma \neq g_j|_\gamma$$

for any Ω_i, Ω_j having common interface γ.

INTERFACES: artificial, with jumps



$g_i = g(\Omega_i) \qquad g_\pm = g(\Omega_\pm)$

From now on chambers are the union of (n)-simplices without metric jumps inside. For technical simplicity we do not consider partly artificial partly real jumps interfaces.

Coordinates

general intrinsic $\{\mathbf{x}\} = (x^1, ..., x^n)$, i.e. any smooth inside Ω.

semi-geodesic $\{\mathbf{q}, \sigma\} = (q^1, ..., q^{n-1}, \sigma)$, $q^\alpha \in \gamma$, $\alpha = 1, .., n-1$, σ- unit outward (inward)normal

Sobolev Space

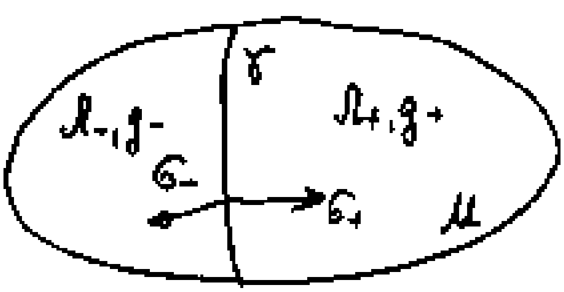
$H_0^1 = \{\Phi \subset H^1(\Omega) : \Phi^+|_\gamma = \Phi_\gamma^-, \Phi|_\Gamma = 0\}$

Laplace-Beltrami operator

The Dirichlet form $\int (\nabla\Phi, \nabla\Phi)$ determines L-B operator by standard procedure of spectral theory

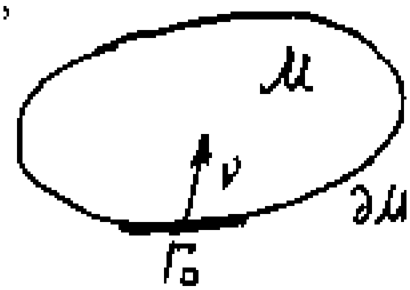
$$\Delta_g \Phi = (|g|)^{-1/2} \frac{\partial}{\partial x^i} (g^{ij} |g|^{1/2} \frac{\partial}{\partial x^j} \Phi)$$

Continuity condition $\begin{cases} \Phi^+|_\gamma = \Phi_\gamma^- \\ \sqrt{g_+} \frac{\partial}{\partial\sigma} \Phi^+|_\gamma = \sqrt{g_-} \frac{\partial}{\partial\sigma} \Phi_\gamma^- \end{cases}$



Boundary Spectral Data (BSD)

$$\begin{cases} \Delta_g \varphi_k(\mathbf{x}) = \lambda_k \varphi_k()\mathbf{x}, & \mathbf{x} \in \mathcal{M}, \\ \varphi_k(\Gamma_0) = 0, & \Gamma_0 \subset \mathcal{M}, \end{cases}$$



BSD:

$$\Gamma_0, \{\lambda_k, \frac{\partial}{\partial\sigma}\varphi_k|_{\Gamma_0}\}_{k=1}^{\infty}$$

Uniqueness Theorem

Consider $\mathcal{M}$, $\tilde{\mathcal{M}}$ - two Admissible Riemannian Polyhedra, such that $\dim \mathcal{M} = \dim \tilde{\mathcal{M}} = n$, then

Th.: Assume BSD ($\mathcal{M}$)=BSD ($\tilde{\mathcal{M}}$), i.e.

$$\begin{cases} \Gamma_0 = \tilde{\Gamma}_0, & \Gamma_0 \subset \partial\mathcal{M}, \tilde{\Gamma}_0 \subset \partial\tilde{\mathcal{M}}, \\ \lambda_k = \tilde{\lambda}_k, \\ \frac{\partial}{\partial\sigma}\varphi_k{}_{\Gamma_0} = \frac{\partial}{\partial\sigma}\tilde{\varphi}_k|_{\tilde{\Gamma}_0} \end{cases}$$

Then polyhedron $\mathcal{M}$ is isometric to polyhedron $\tilde{\mathcal{M}}$.

Published Results

Katchalov A., Kurylev Ya., Lassas M., "**Inverse Boundary Spectral Problems**", 2001

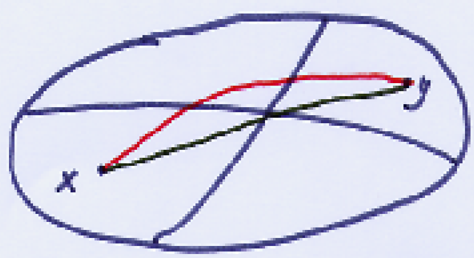
- Uniqueness results for smooth Riemannian manifolds
- Reconstruction procedure from BSD given on a part of the manifold boundary
- Equivalence of spectral and dynamic data
- BC-method (Belishev M.)

Difficulties

Trying to apply methods from the KKL book we have encounted the following difficulties (presence of interfaces and conical points gave rise to them):

- boundary distance functions are not one-to one
- length space to be defined
- Gaussian beams on interfaces for multi-D anisotropic case
- Tataru's type uniqueness theorems
- Kervaire, 1960 constructed 10D polyhedron which does not admit any differentiable structure
- separating properties of eigenfunctions are not known at conical points

Eells., Fuglede B., "**Harmonic maps between Riemannian Polyhcdra**", 2001



Kirpichnikova A., "**The behaviour of the Gaussian beam in the anisotropic medium with an interface**", 2005

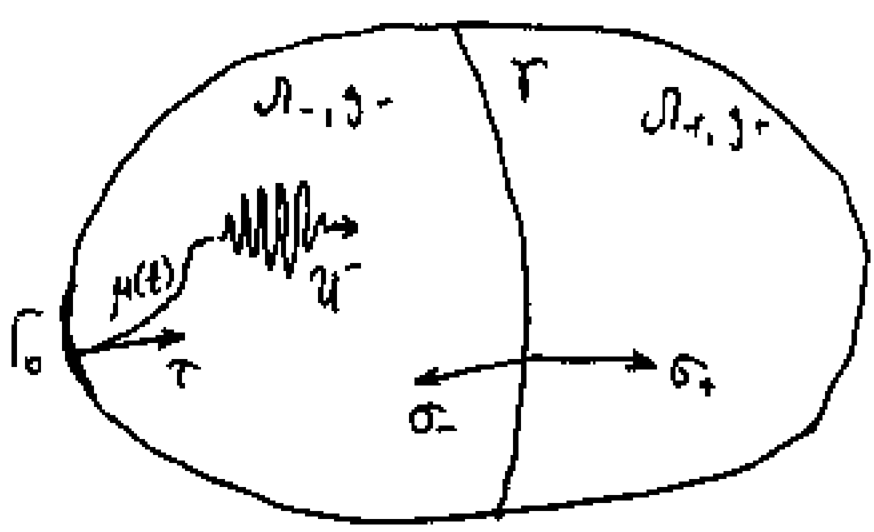
Gaussian Beams

are special solutions to the WE

$$\begin{cases} \Box_g U = 0, \\ U|_{t<t_0} = \partial_t U|_{t<t_0} = 0, \\ U|_{\Gamma_0} = f^0(\varepsilon; t, z) = M_\varepsilon \chi \exp\{i\varepsilon^{-1}\Theta\Psi^0(z)\}, \\ U^- [illegible] = U^+ [illegible], \\ \sqrt{g^-}\frac{\partial}{\partial\sigma}U^- [illegible] = \sqrt{g^+}\frac{\partial}{\partial\sigma}U^+ [illegible], \end{cases}$$

- propagate along the geodesic $\mu(t)$ with unit velocity,

- are concentrated near the point on $\mu(t)$ at time t.



Mathematically Gaussian beams can be found in the form of formal series w.r.t. small parameter $0 < \varepsilon < 1$:

$$U \approx M_\varepsilon \cdot \exp\{\ i\varepsilon^{-1}\Theta\}\sum_{l\geq 0} u_l (i\varepsilon)^l,\ M_\varepsilon = (\pi\varepsilon)^{\frac{3}{4}}$$

The physical properties require phase function Θ to be real at one point on $\mu(t)$ which propagates along the geodesic at time t and positive definite everywhere else:

$$\begin{cases} \Im\Theta(\mu(t)) = 0, \\ \Im\Theta(x) > 0,\ x \neq \mu(t). \end{cases}$$

At any time t the concentration of energy of a Gaussian beam coincides with Gaussian distribution (error function) hence the name of the WE solution. They are also called "**quasi-photons**".

Phase Function

$$\Theta \asymp p_i(t)(x^i - \mu^i(t)) + \frac{1}{2}H_{ij}(x^i - \mu^i(t))(x^j - \mu^j(t)) + ...$$

Conditions on Θ to be a phase function of a Gaussian beam require

$$\Im p(t) = 0, \quad \Im H(t) \geq 0.$$

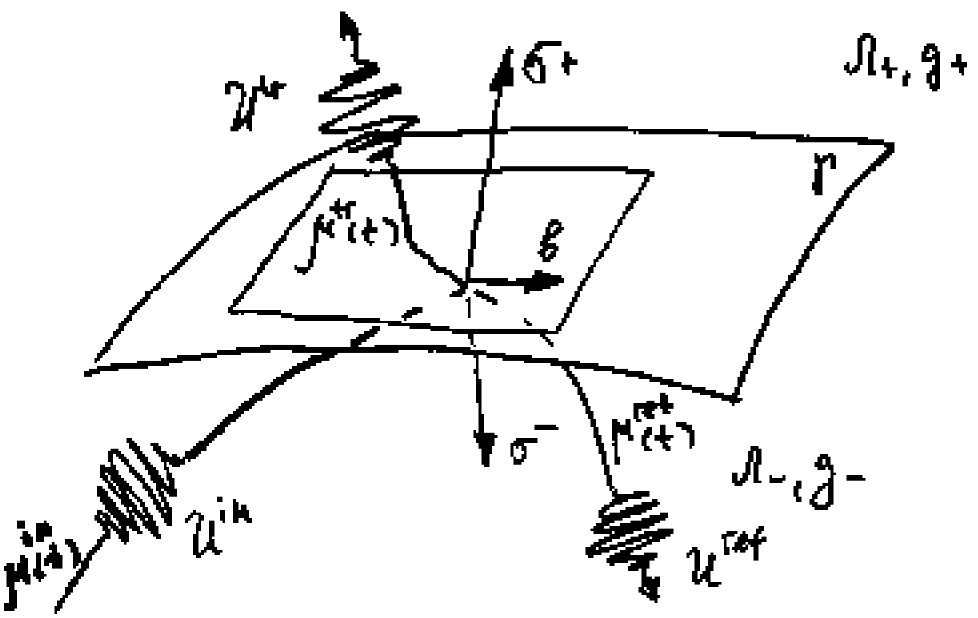
The corresponding hamiltonian

$$h^2 = g^{\alpha\beta} p_\alpha(t) p_\beta(t) + (p_n(t))^2.$$

Each term of the phase function decomposition (as well as the amplitude functions) can be found as a solution of some first order differential equations, supplied with appropriate initial data. The basic geodesic and impulses are solutions of **Hamilton (canonical) equations**, the second term - quadratic form $H(t)$ is a solution of a **Riccati equation** and so on.

Gaussian Beams on interface

Using the results of Eells and Fuglede we can concentrate on Gaussian beams not going through conical points and consider their behaviour on interfaces.



The total field consists of three components: U^{in}, U^{ref}, U^{tr}. Continuity conditions + the fact that each U is a solution of WE → give conditions on each term of Θ (and amplitude function u).

$$\Theta^{in}|_{\gamma} = \Theta^{ref}|_{\gamma} = \Theta^{tr}|_{\gamma}.$$

Careful calculations allow us to find the coefficients for all terms of Θ and u hence we find a Gaussian beam.

Transmission and Reflection Laws

This means that $\nabla\Theta$ and p lie in the same $2D$ **coplane**, there are standard reflection and transmission angles in this coplane,

$$p = (\mathbf{b}, p_n),\; b := d\Theta^{in}|_\gamma = d\Theta^{ref}|_\gamma = d\Theta^{tr}|_\gamma$$

The linear part leads us to **Anisotropic generalization of Snell's Law:**

$$\begin{cases} \sin\varphi^{tr} = \sin\varphi^{in}\sqrt{\frac{g_-^{\alpha\beta} b_\alpha b_\beta}{g^{\alpha\beta} b_\alpha b_\beta}}, \\ \cos\varphi^{tr} = \sqrt{1 - \sin\varphi^{tr\,2}} \end{cases}$$

Anisotropic generalization of Frenel's Law:

$$\begin{cases} \sin\varphi^{ref} = sin\varphi^{in} < 1, \\ \cos\varphi^{ref} = -\cos\varphi^{in} > 0, \end{cases}$$

For $2D$ coplane there exists a critical angle φ^{in} where $\varphi^{tr} = \frac{\pi}{2}$. For our inverse problem we consider only $\varphi^{in} \sim 0$ and $0 \leq g_+^{\alpha\beta} p_\alpha^{in} p_\beta^{in} < 1$.

The main result we need is that once we have a metric jump the energy will reflect in at least some plane.

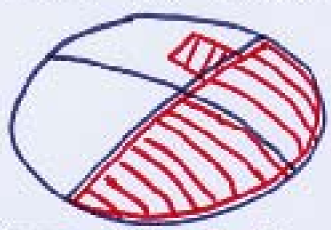
Sketch of the Uniqueness Th. Proof

Consider two ARP $\mathcal{M}$ and $\tilde{\mathcal{M}}$ with equal(diffeo) BSD. We start reconstruction from the parts Γ_0, $\tilde{\Gamma}_0$ of ARP boundaries by BC-method. On each stage we extend the known data region and show that the obtained regions are isometric on both ARP.

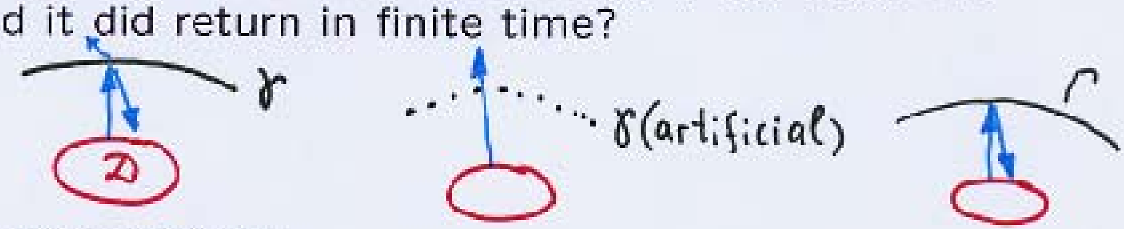
Start We assume that we know the data on some part of the boundary $\Gamma_0 \subset \partial\mathcal{M}$, $\tilde{\Gamma}_0 \subset \tilde{\mathcal{M}}$. Using procedure from KKL book, we reconstruct the first regions D, $\tilde{D}$, isometric to each other.

Data recalculation We can recalculate BSD on ∂D, $\partial\tilde{D}$ and show that they are equal (see KKL)

Extending to whole chamber Ω_1 We continue previous procedure.



Going through the interface What if we sent GB and it did return in finite time?



Whole Polyhedron