Geometry and Analysis of the Faddeev model

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D. Auckly, L. Kapitanski. *Holonomy and Skyrme's model*, Comm. Math. Phys., **240**, 97-122 (2003)

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L. D. Faddeev: 1975

particles – spatially localized solutions of PDEs
with enough bells and whistles
(charge, etc.)
+ have internal knotted structure

motivation: **Skyrme's model** (1961):
particles – spatially localized solutions of PDEs
with enough bells and whistles
(Lord Kelvin's "vortex atoms", 1867)

In the original **Skyrme model**:

fields are **maps** $\mathbb{R}^3 \to S^3$ with $\{|x| = \infty\} \mapsto \mathbf{1}$

homotopy classes of such maps are classified by **degree** (called "topological charge", "baryon number", etc. - "bells and whistles")

$S^3 \simeq SU(2) \simeq$ unit quaternions

$$u = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \qquad |z_1|^2 + |z_2|^2 = 1$$

$$z_1 = u_0 + u_1\,\mathbf{i}, \qquad z_2 = u_2 + u_3\,\mathbf{i}$$

$$u = u_0 + u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$$

$$u^{-1} = u^* = u_0 - u_1\,\mathbf{i} - u_2\,\mathbf{j} - u_3\,\mathbf{k}$$

$\mathbb{R}^3 \simeq su(2) \simeq$ purely imaginary quaternions

$$a = \begin{pmatrix} a_1\mathbf{i} & a_2 + a_3\mathbf{i} \\ -a_2 + a_3\mathbf{i} & -a_1\mathbf{i} \end{pmatrix}$$

$$a = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$$

$$\langle a,\, b\rangle = \vec{a}\cdot\vec{b} = -\tfrac{1}{2}\,\mathrm{Trace}(ab) = \mathrm{Re}(a\,b^*)$$

Skyrme: $u : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$

Skyrme energy (static Hamiltonian):

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2}|du|^2 + \frac{1}{4}|du \wedge du|^2 \quad dx$$

Topological charge:

$$Q(u) = c \int_{\mathbb{R}^3} \sum \epsilon_{\alpha\beta\gamma\delta} \; u^\alpha \, \frac{\partial\left(u^\beta, u^\gamma, u^\delta\right)}{\partial\left(x^1, x^2, x^3\right)} \, d^3x$$

Skyrme: $u : \mathbb{R}^3 \to SU(2)$ $\boxed{a = u^{-1}du}$

– $su(2)$-valued 1-form – flat connection – pull-back of the Maurer-Cartan form $g^{-1}dg$

Skyrme energy (static Hamiltonian):

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2}|u^{-1}\, du|^2 + \frac{1}{4}|u^{-1}\, du \wedge u^{-1}\, du|^2$$

$$E[a] = \int_{\mathbb{R}^3} \frac{1}{2}|a|^2 + \frac{1}{16}|[a,\, a]|^2$$

Topological charge:

$$Q(u) = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \mathrm{Tr}\left(u^{-1}\, du \wedge u^{-1}\, du \wedge u^{-1}\, du\right)$$

$$Q[a] = c \int_{\mathbb{R}^3} \langle a,\ [a,\, a]\rangle$$

$$E[a] \geq \text{const}\ |Q[a]|$$

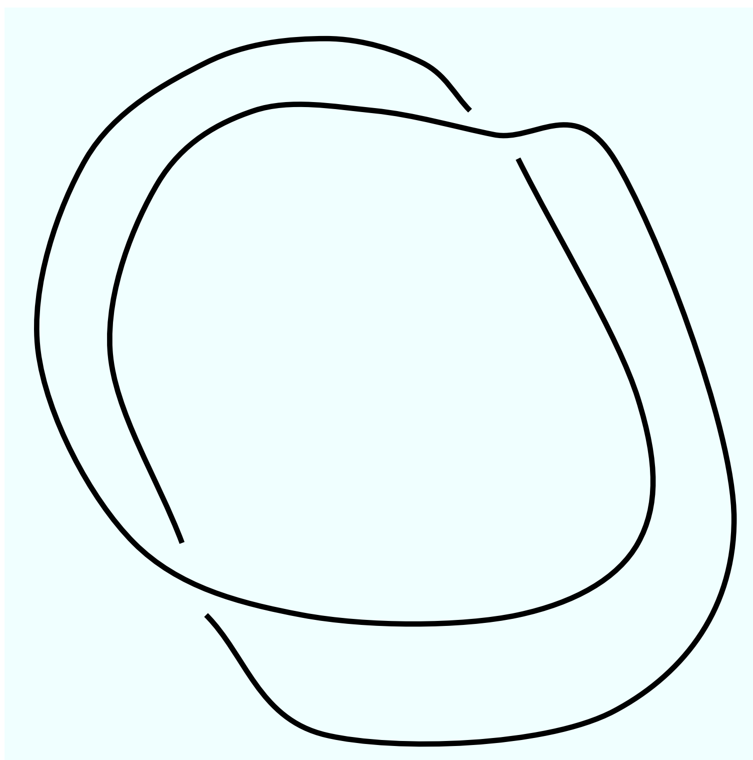
In the original **Faddeev** model:

fields are **maps** $\mathbf{n} : \mathbb{R}^3 \to S^2 \subset \mathbb{R}^3$
with $\{|x| = \infty\} \mapsto$ pole

homotopy classes of such maps are classified by the **Hopf invariant** (called "topological charge", "linking number", etc.)

Hopf invariant

$$S^3 \rightarrow S^2$$



Faddeev fields: $\mathbf{n} : \mathbb{R}^3 \to S^2 \subset \mathbb{R}^3$

$$\mathbf{n}(x) = (n^1(x), n^2(x), n^3(x)) \qquad |\mathbf{n}(x)| = 1$$

Faddeev energy: $E(\mathbf{n}) = \int\limits_{\mathbb{R}^3} |d\mathbf{n}|^2 + |d\mathbf{n} \wedge d\mathbf{n}|^2$

$$|d\mathbf{n}|^2 = \frac{\partial n^a}{\partial x^k} \frac{\partial n^a}{\partial x^k}$$

$$|d\mathbf{n} \wedge d\mathbf{n}|^2 = \sum_{i,j} \left| \frac{\partial \vec{n}}{\partial x^i} \times \frac{\partial \vec{n}}{\partial x^j} \right|^2 = \sum_{i,j} \left(\vec{n}, \frac{\partial \vec{n}}{\partial x^i}, \frac{\partial \vec{n}}{\partial x^j} \right)^2$$

Hopf number: $Q(\mathbf{n}) = \int\limits_{\mathbb{R}^3} \alpha \wedge d\alpha \ \in \mathbb{Z}$

$$d\alpha = \mathbf{n}^* \omega_{S^2} \qquad \delta\alpha = 0$$

$$\omega_{S^2} = \frac{1}{4\pi} \left(n^1 dn^2 \wedge dn^3 + n^2 dn^3 \wedge dn^1 + n^3 dn^1 \wedge dn^2 \right)$$

Estimate: $E(\mathbf{n}) \geq c\,|Q(\mathbf{n})|^{3/4}$

A. F. Vakulenko & L. V. Kapitanski "Stability of Solitons in S^2-Nonlinear σ-Model", Sov. Phys. Doklady, **24** (6) (June 1979), 443-444

Existence of minimizers:
For $Q(\mathbf{n}) = 1$ and an infinite number of other possible values:

Lin, Fanghua; Yang, Yisong "Existence of energy minimizers as stable knotted solitons in the Faddeev model." Comm. Math. Phys. 249 (2004), no. 2, 273–303

When $\mathbb{R}^3$ or S^3 is replaced by a general Riemannian three-manifold, M^3, the homotopy classification of maps to S^2 is more complicated.

Theorem [Pontrjagin, 1941] Let M be a closed, connected, oriented three-manifold. To any continuous map φ from M to S^2 one associates a cohomology class $\varphi^*\mu_{S^2} \in H^2(M;\mathbb{Z})$, where μ_{S^2} is a generator of $H^2(S^2;\mathbb{Z})$. Every class may be obtained from some map, and two maps with different classes lie in different homotopy classes. The homotopy classes of maps with a fixed class $\alpha \in H^2(M;\mathbb{Z})$ are in bijective correspondence with
$H^3(M;\mathbb{Z})/(2\,\alpha \cup H^1(M;\mathbb{Z}))$.

New features:
1) There is a new invariant given by the induced map on second cohomology.
2) The Hopf invariant generalizes into a secondary invariant that sometimes takes values in a finite cyclic group.

Example. smooth $\varphi : T^3 \to S^2$

Let γ_1, γ_2, γ_3 be a basis in $H_1(T^3, \mathbb{Z})$.
The inverse image of a regular value $p \in S^2$ is a curve, $\gamma = \varphi^{-1}(p)$, and
$\gamma \sim m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3$ in $H_1(T^3, \mathbb{Z})$.
These m_1, m_2, m_3 are homotopy invariants.

Find $m =$ g.c.d.(m_1, m_2, m_3).

Case $m = 0$: There are $\mathbb{Z}$ different homotopy classes, distinguished by Hopf number.

Case $m \neq 0$: There are $2m$ different homotopy classes corresponding to the same m_1, m_2, m_3.

A simpler **example:** $\varphi : S^2 \times S^1 \to S^2$

$(z, \theta) \mapsto \varphi(z, \theta)$; $\varphi(\cdot, \theta) : S^2 \to S^2$

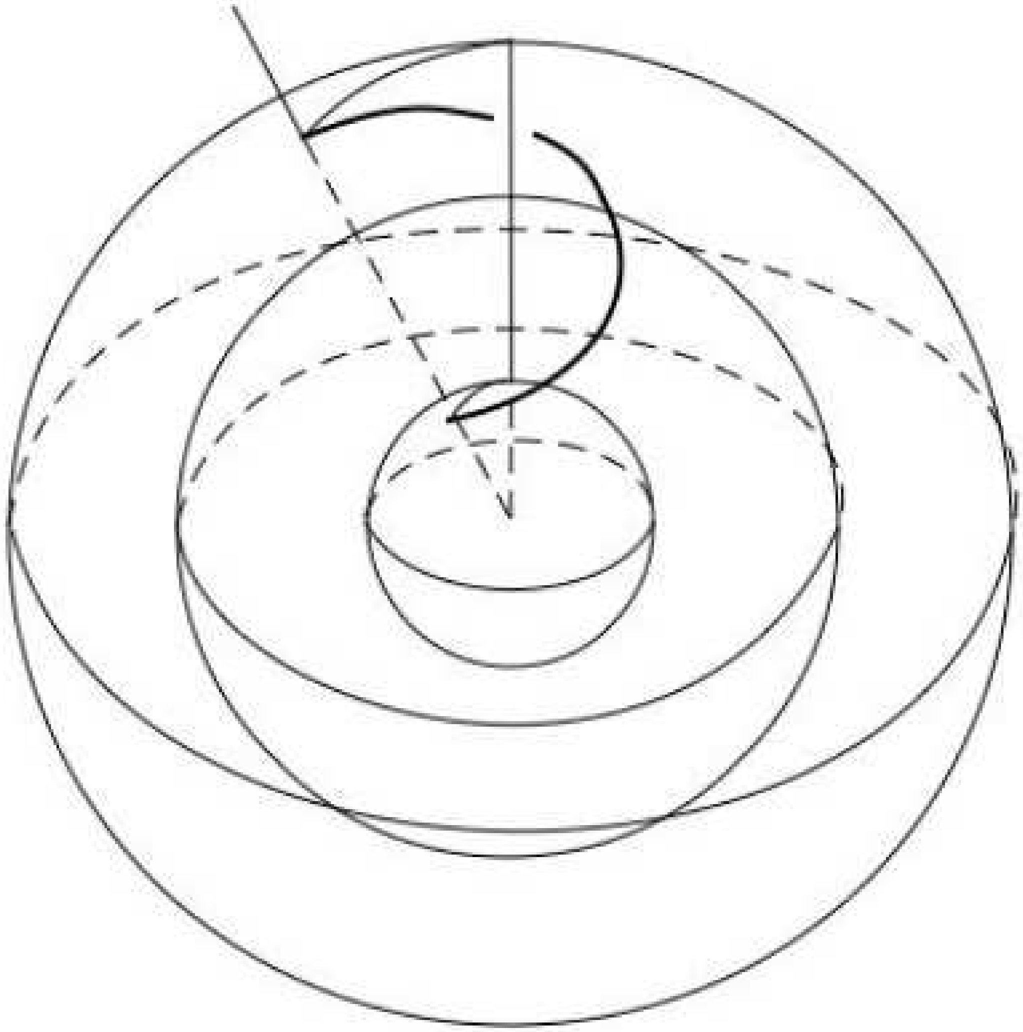
The primary invariant: $m = \text{degree}\,(\varphi(\cdot, \theta))$

If $m \neq 0$, there are $2m$ different homotopy classes corresponding to the same degree.

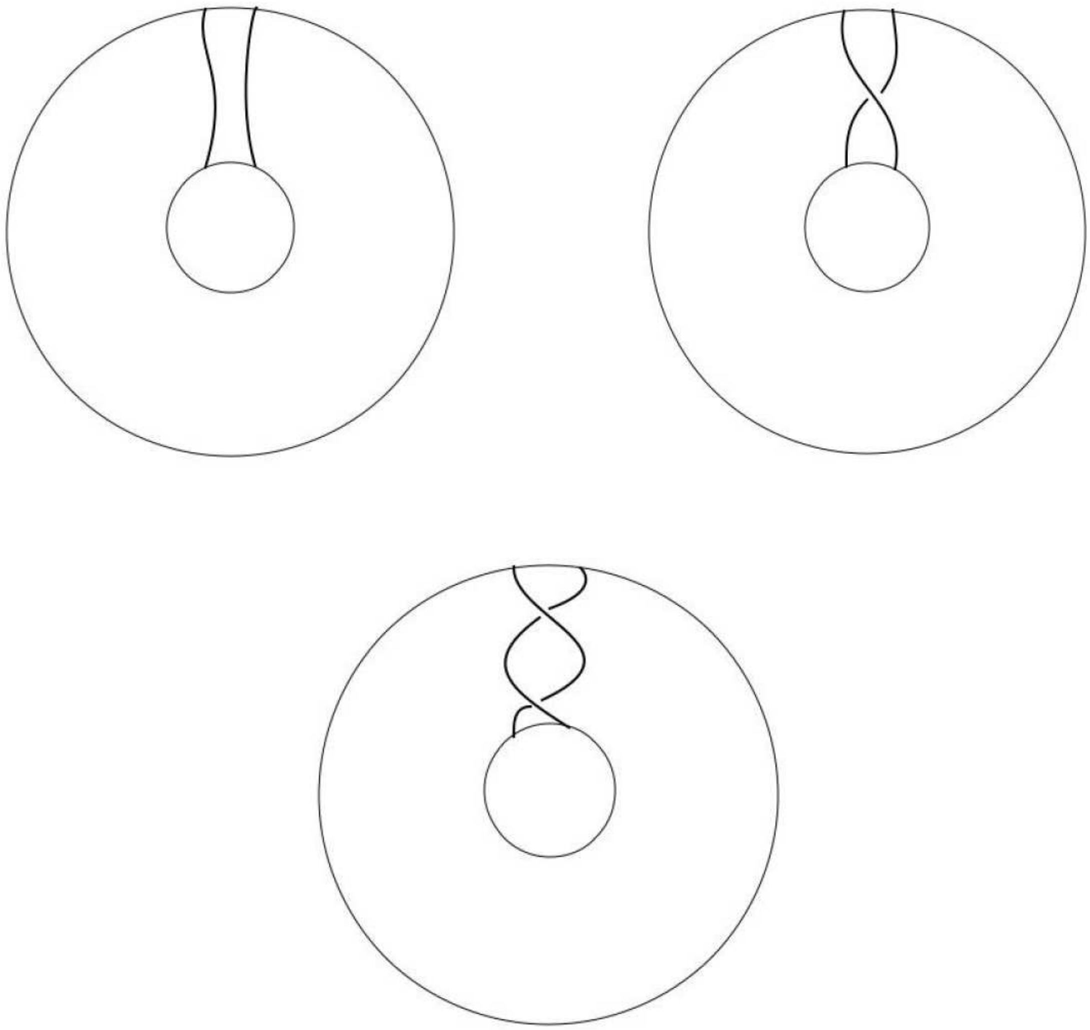
To visualize $m = 1$ case: 2 homotopy classes:

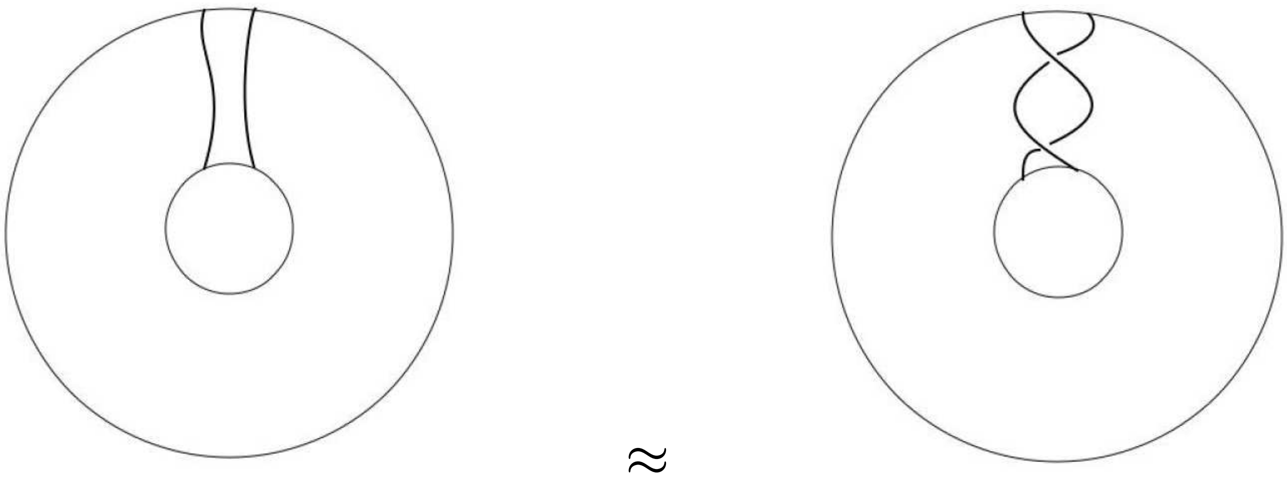
1) $(z, \theta) \mapsto z$

2) $(z, \theta) \mapsto z\, e^{\mathbf{i}\theta}$



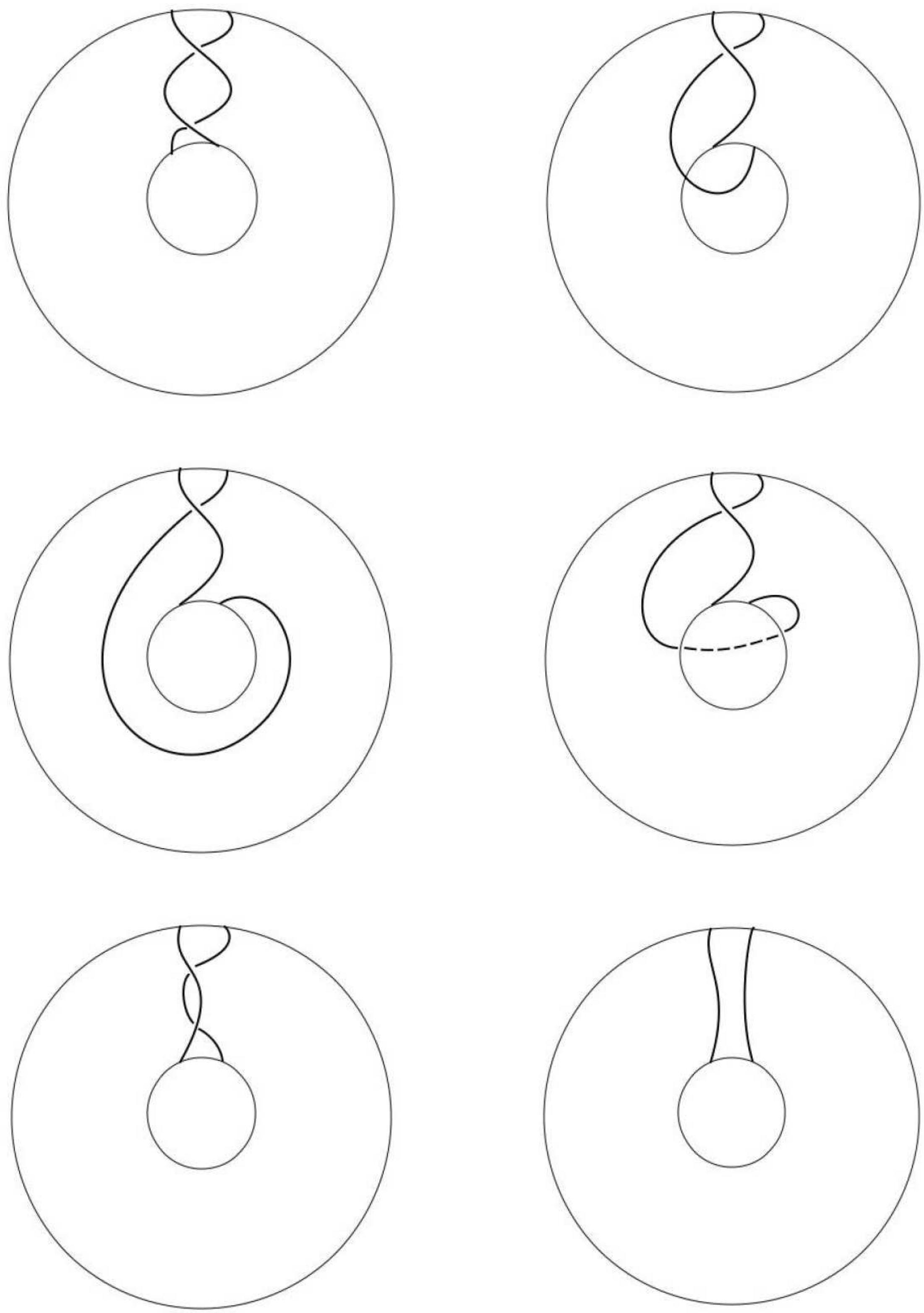
The pictures that follow illustrate the first half of the map, i.e., $S^2 \times [0, \pi] \to S^2$, i.e., between the innermost and the middle spheres.





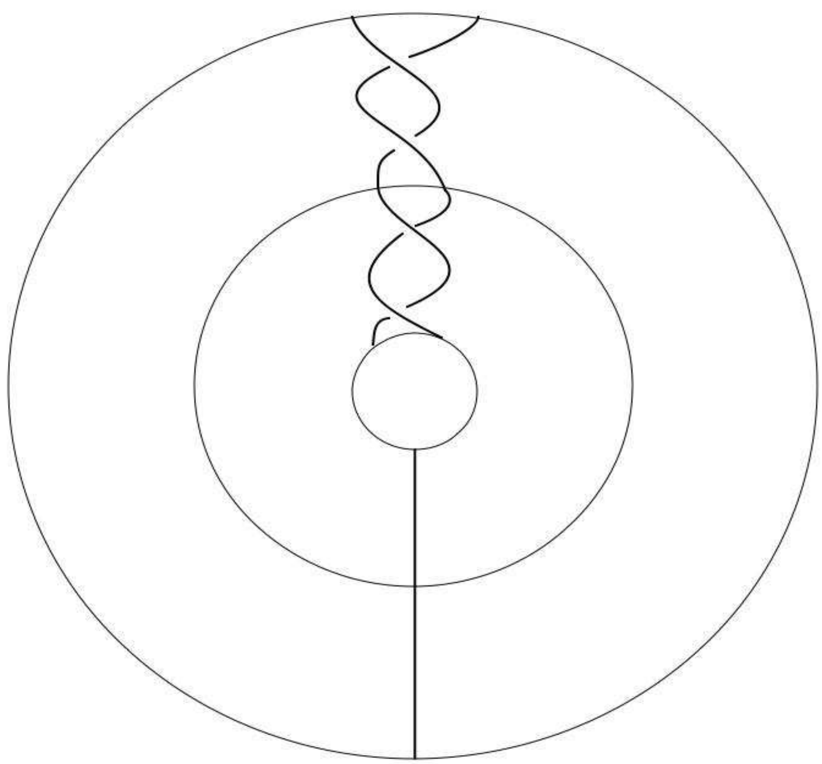
Dirac's strings problem

[one half of the full picture, see later]

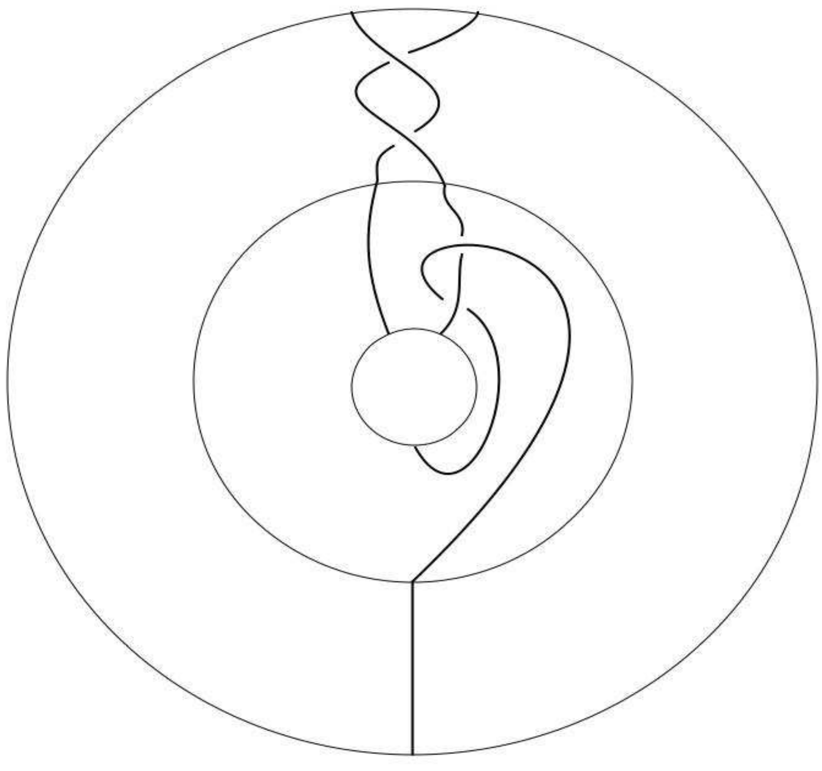


Putting two halves together:

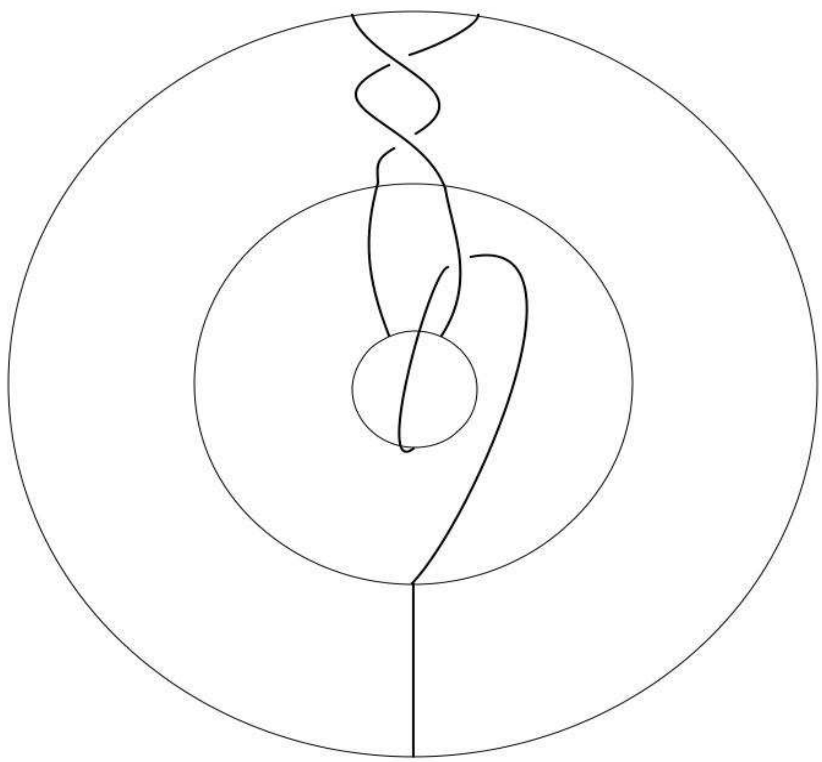
Start with



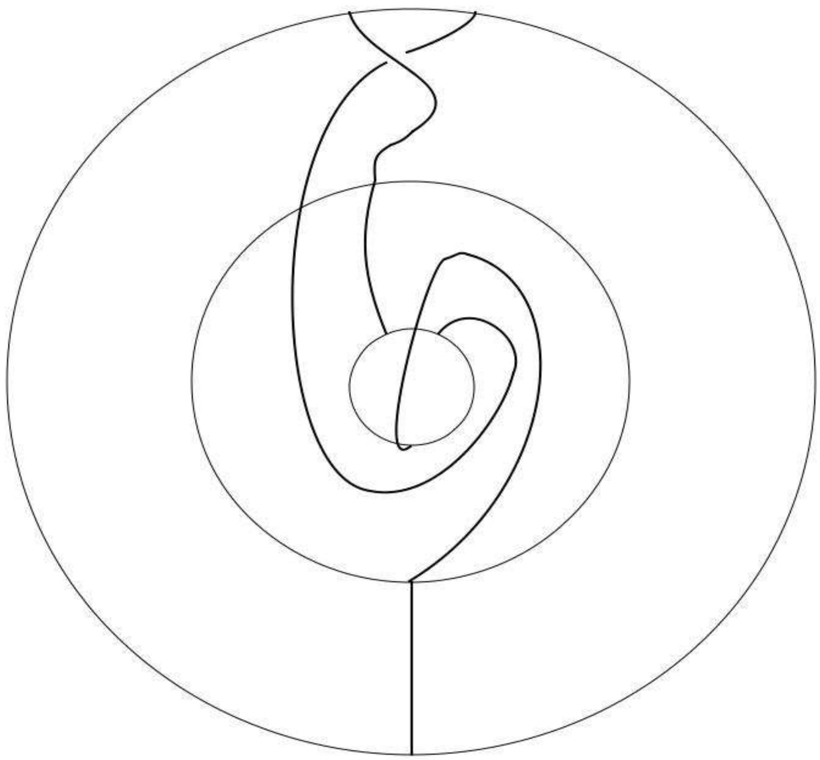
After "one half" has been changed:



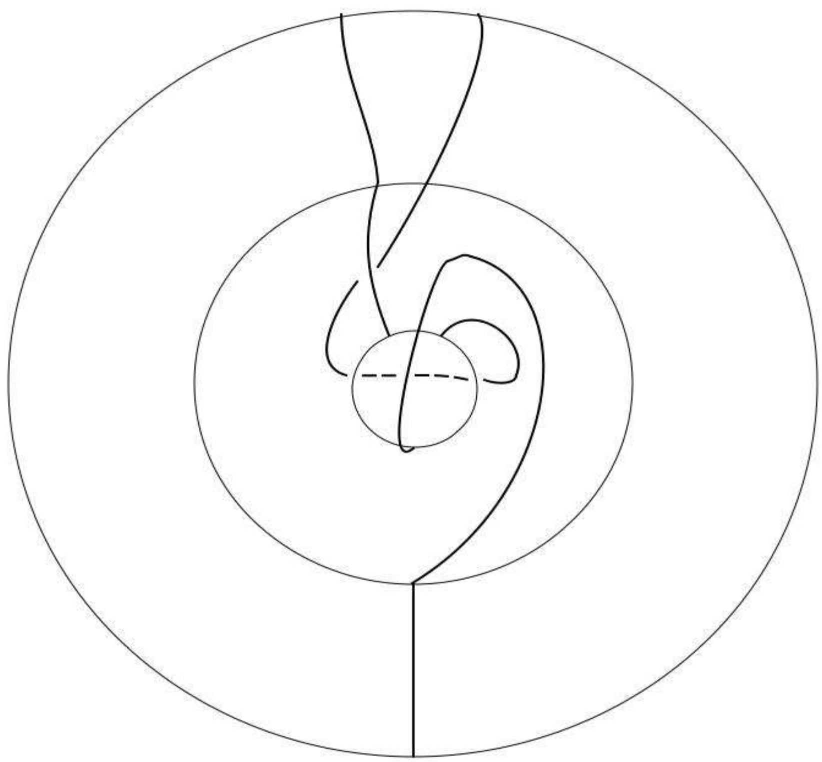
i.e.,



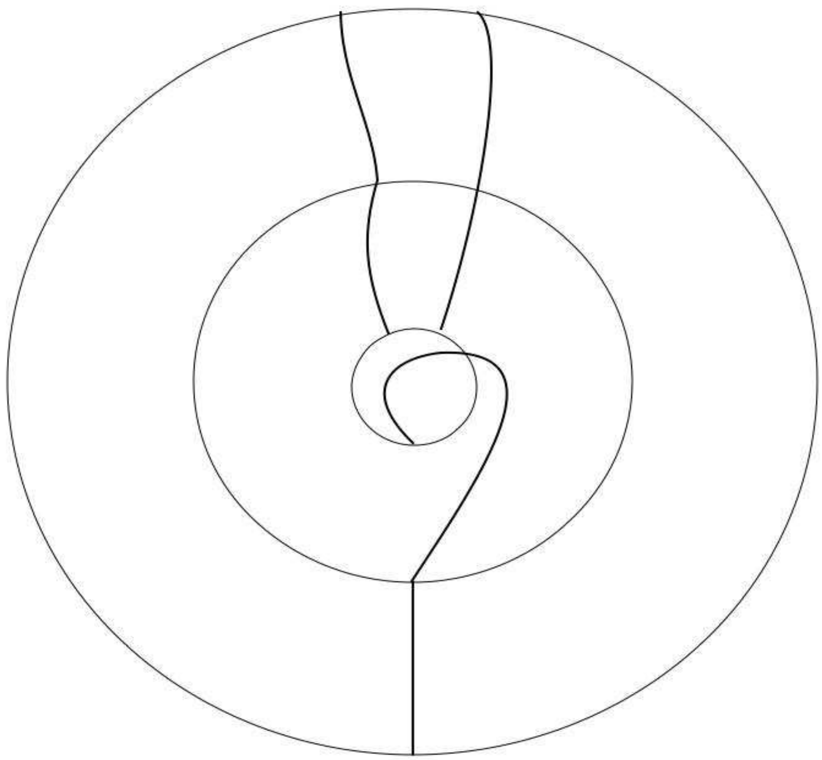
Now repeat the maneuver:



and



finally,



S^3 – unit quaternions: $q = q_0 + q_1\,\mathbf{i} + q_2\,\mathbf{j} + q_3\,\mathbf{k}$
S^2 – unit sphere in the purely imaginary quaternions
S^1 – unit complex numbers $\subset S^3$

Hopf map: $h: S^3 \to S^2 \qquad q \mapsto q\,\mathbf{i}\,q^{-1}$

Dirac's strings: $\mathfrak{q}: S^2 \times S^1 \to S^3$
$\mathfrak{q}(z, \lambda) = q\lambda q^{-1}$, where $z = q\,\mathbf{i}\,q^{-1}$
$\deg \mathfrak{q} = 2$.

Lemma. Given φ, ψ : $M \to S^2$, there exists a map $u : M \to S^3$ such that $\psi = u\,\varphi\,u^{-1}$ iff $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$.
If $\psi = u\,\varphi\,u^{-1}$, then $\psi = \tilde{u}\,\varphi\,\tilde{u}^{-1}$,
where $\tilde{u} = u\,\mathfrak{q}(\varphi, \lambda)$, and λ is any map $M \to S^1$.

$$\mathfrak{q}(\varphi,\lambda)^* \mu_{S^3} = (\varphi,\lambda)^* \mathfrak{q}^* \mu_{S^3} = (\varphi,\lambda)^* (2\mu_{S^2} \cup \mu_{S^1})$$
$$= 2\,\varphi^* \mu_{S^2} \cup \lambda^* \mu_{S^1}$$

Fix $\varphi : M \to S^2$. The map $\eta \mapsto (\varphi^* \mu_{S^2} \cup \eta)[M]$ from $H^1(M; \mathbb{Z})$ to $\mathbb{Z}$ is a group homomorphism, hence has image $m\mathbb{Z}$ for some m depending on the class $\varphi^* \mu_{S^2}$.

Theorem (Auckly & K). All homotopy classes of maps $\psi : M \to S^2$ with the same second cohomology class $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$ are obtained in

the form $\psi = u\,\varphi\,u^{-1}$.

The maps $u_1\,\varphi\,u_1^{-1}$ and $u_2\,\varphi\,u_2^{-1}$ are homotopic if and only if

$$\deg u_1 \equiv \deg u_2 \,(\mathrm{mod}\ 2\,m)$$

Sketch of a piece of the proof

Čech picture: do locally, then patch together

1) What does $\varphi^* \mu_{S^2}$ mean?

2) How can one find $u : M^3 \to S^3$ such that $\psi = u\,\varphi\,u^{-1}$?

Local Representation. If $\varphi : I^3 \to S^2$, then there exists $u : I^3 \to S^3$ such that $\varphi = u^{-1}\,\mathbf{i}\,u$. For any two such maps, u and v, there is a map $\lambda : I^3 \to S^1$ so that $v = \lambda u$.

Assume for a moment that we knew that such a map existed. Then

$$\varphi^{-1} d\varphi = a + \varphi a \varphi, \qquad \text{where} \quad a = u^{-1} du$$

Hence,

$$a = \frac{1}{2}\varphi^{-1} d\varphi + \varphi\xi$$

for some real valued 1-form ξ. Since a is flat,

$$0 = da + a \wedge a = \varphi d\xi - \frac{1}{4} d\varphi \wedge d\varphi$$

or, equivalently,

$$d\xi = -\frac{1}{4}\varphi d\varphi \wedge d\varphi$$

We will turn this around by solving for ξ, then a, and, finally, u.

One can directly check that $\varphi d\varphi \wedge d\varphi$ is real and closed. By the Poincaré lemma, there exists a 1-form ξ such that $d\xi = -\frac{1}{4}\varphi d\varphi \wedge d\varphi$.

Set $a = \frac{1}{2}\varphi^* d\varphi + \varphi\xi$. This is an $su(2)$-valued 1-form, and it is flat: $da + a \wedge a = 0$. By the *nonlinear* Poincaré lemma, there exists a $w : I^3 \to S^3$ with $a = w^{-1}dw$.

Consider $\psi = w\varphi w^{-1}$.
Since $\varphi(x) \in S^2$, $\psi(x) \in S^2$ as well. Moreover, $\psi(x) \equiv z = \text{const} \in S^2$. Indeed,

$$\psi^{-1}d\psi = w(\varphi^{-1}d\varphi - a - \varphi a \varphi)w^{-1} = 0$$

The Hopf map, $h : S^3 \to S^2$, is onto, hence there is a $p \in S^3$ so that $w\varphi w^{-1} = p^{-1}\mathbf{i}p$. Take $u = pw$ to get $\varphi = u^{-1}\mathbf{i}u$.

Nonlinear Poincaré Lemma (Auckly & K.) Given any L^2 $\mathfrak{g}$-valued 1-form A on I^m such that

$$dA + \frac{1}{2}[A, A] = 0 \tag{1}$$

in the sense of distributions, there exists $u \in W^{1,2}(I^m, G)$ such that $u^{-1} \in W^{1,2}(I^m, G)$ and $A = u^{-1}\, du$. Furthermore, for any two such maps, u and v, there exists $g \in G$ so that $u(x) = g \cdot v(x)$, for almost every $x \in I^m$.

maps $\leftrightarrow$ **connections**

$$u : M \to S^3 \qquad \rightsquigarrow \qquad a = u^{-1}\, du$$

$$\begin{aligned} \deg u &= -\frac{1}{12\pi^2} \int_M \mathrm{Re}(a \wedge a \wedge a) \\ &= \frac{1}{4\pi^2} \int_M \mathrm{Re}(a \wedge da + \frac{2}{3} a \wedge a \wedge a) = \mathrm{cs}(a) \end{aligned}$$

$$\tilde{u} = u\, \mathfrak{q}(\varphi, \lambda) \quad \rightsquigarrow \qquad \tilde{a} = \tilde{u}^{-1}\, d\tilde{u}$$

Varying $\lambda : M \to S^1$, . . .

Theorem. Any orientation preserving S^2-isometry class of a smooth map from M to S^2 homotopic to φ is uniquely represented by a smooth flat connection a, which has trivial holonomy and satisfies the conditions

1. $\mathrm{cs}(a) = -\frac{1}{12\pi^2} \int_M \mathrm{Re}(a \wedge a \wedge a) \in 2m\mathbb{Z}$

2. $\mathcal{H}\langle a, \varphi\rangle = h_1 \eta_1 + \cdots + h_b \eta_b$
   with $h_1, \ldots, h_b \in [0, 1)$ and
   $\eta_1, \ldots, \eta_b$ – an integral basis for $H^1(M; \mathbb{R})$

3. $\delta \langle a, \varphi\rangle = 0$ (δ is the adjoint of d)

Faddeev energy of $\psi : M \to S^2$ is

$$E(\psi) = \int_M |d\psi|^2 + |d\psi \wedge d\psi|^2$$

If ψ is smooth and homotopic to φ,
then $\psi = u\varphi u^{-1}$. Re-write $E(\psi)$ in terms of φ and $a = u^{-1}du$

$$E(\psi) = E_\varphi[a] = \int_M |D_a\varphi|^2 + |D_a\varphi \wedge D_a\varphi|^2$$

where $D_a\varphi = d\varphi + [a, \varphi]$

Class $\mathfrak{A}_\varphi$: $a \in L^2(M; \mathbb{R}^3)$, $da + a \wedge a = 0$,
$\rho_a = 0$, $E_\varphi[a] < \infty$, $\mathrm{cs}(a) \in 2m\mathbb{Z}$,
$\mathcal{H}\langle a, \varphi\rangle = h_1 \eta_1 + \cdots + h_b \eta_b$, $h_1, \ldots, h_b \in [0, 1]$,
$\delta \langle a, \varphi\rangle = 0$

Theorem. $E_\varphi[a]$ has a minimum in $\mathfrak{A}_\varphi$.