Structure of the spectrum of acoustic operator with singularly pertubed periodic coefficients

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$$-\mathbb{L}u(x) := \nabla a(x) \nabla u(x) + q(x) u(x),$$
$$x \in \mathbb{R}^n$$

$$a(x), q(x) > 0$$
$$a(x+2\pi m) = a(x), \; q(x+2\pi m) = q(x),$$
$$m \in \mathbb{Z}^n$$
$$\sigma(\mathbb{L}) - ?$$

The answer is given by Floquet theory

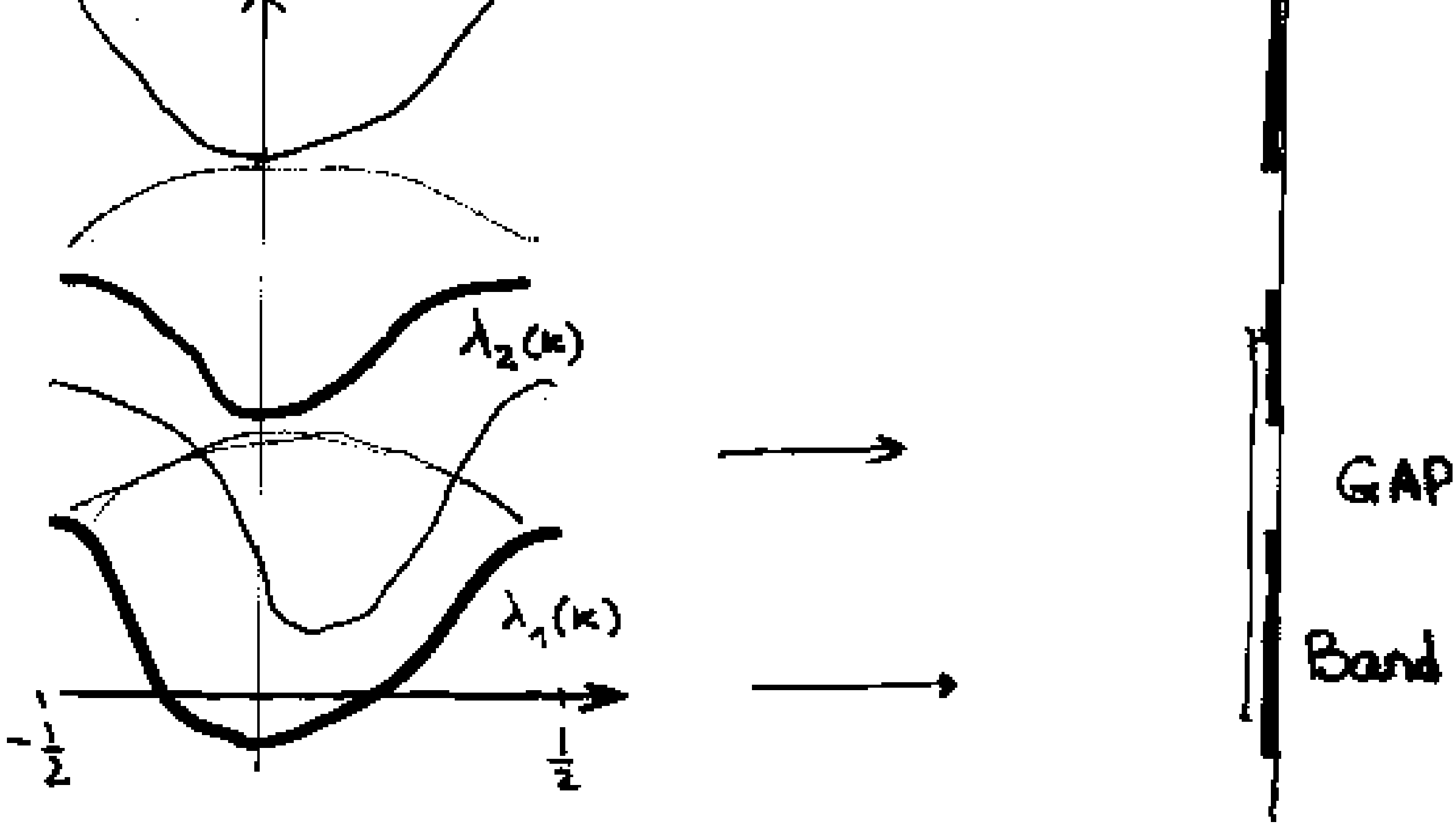
$$L(\kappa) = -\nabla a \nabla + q \quad , \qquad [-\pi, \pi]^n$$

$$u(x+2\pi m) = e^{2\pi i \kappa m} u(x), \; m \in \mathbb{Z}^n$$

$L(\kappa)$ - self-adjoint, with compact resolvent

$\exists \; \lambda_n(\kappa), u_n(x,\kappa) \quad , \quad \lambda_n(\kappa) \leq \lambda_{n+1}(\kappa) \to +\infty$

$$L(\kappa) u_n = \lambda_n(\kappa) u_n$$
$$\kappa \in [-\tfrac{1}{2}, \tfrac{1}{2}]^m$$



$$\sigma(\mathbb{L}) = \bigcup_{k \in (-\frac{1}{2}, \frac{1}{2}]} \sigma(L(k)) = \bigcup_{m} [\min_{k} \lambda_n(k), \max \lambda_n(k)$$

Photonic fibers

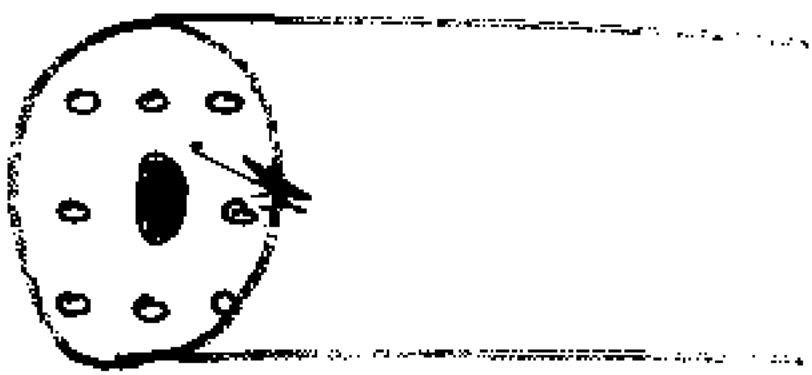
(2+1)D -periodic Maxwell operator

$$\begin{cases} \nabla \times E = -i\frac{\omega}{c}\mu H, & \nabla \cdot \mu H = 0 \\ \nabla \times H = i\frac{\omega}{c}\varepsilon E, & \nabla \cdot \varepsilon E = 0 \end{cases}$$

$\varepsilon(x), \mu(x)$ electric and magnetic permeabilities

$\mu(x) = \text{const}$

$\varepsilon(x) = \varepsilon(x_1, x_2)$ 2D-periodic



$$E(\underline{x}) = e^{ik_3x_3}\,\mathcal{U}(x_1, x_2)$$

$$H(\underline{x}) = e^{ik_3x_3}\,V(x_1, x_2)$$

No rigorous results.

Exeption: $k_3 = 0$

$$\underline{E}(\underline{x}) = (0, 0, E(x_1, x_2))$$

$$\underline{H}(\underline{x}) = (0, 0, H(x_1, x_2))$$

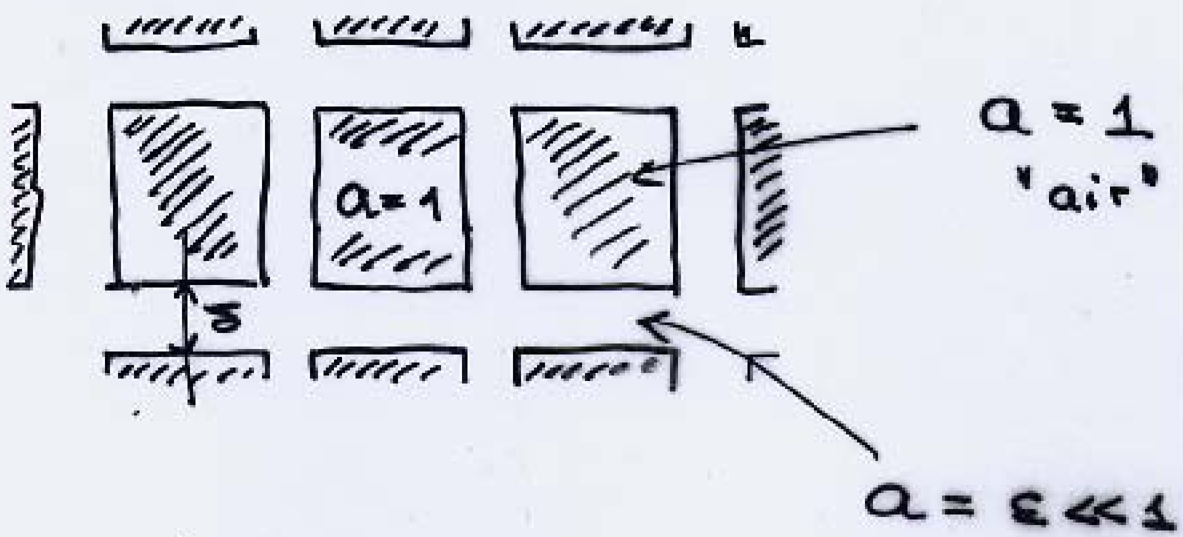
$$-\Delta E = \lambda \varepsilon(x) E$$

$$-\nabla \frac{1}{\varepsilon(x)} \nabla H = \lambda H$$

scalar equations.

Figotin, Kuchment 1996

$$-\nabla a(x) \nabla H = \lambda H$$



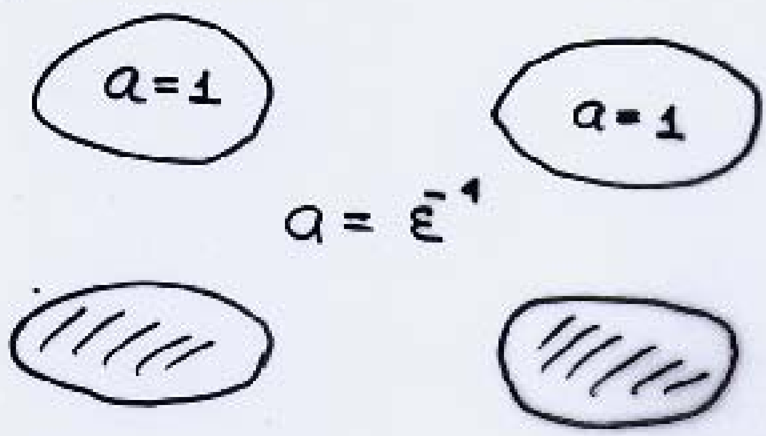
$\delta, \delta^{-1}\varepsilon$ and $\frac{\delta^2}{\varepsilon} \to 0 \Rightarrow$ GAP

- Reduction to 1D case

Variational approach

Hempel, Lienau 2000

$-\nabla a \nabla u = \lambda u$



$\varepsilon \to 0 \quad \Rightarrow \quad$ GAPS

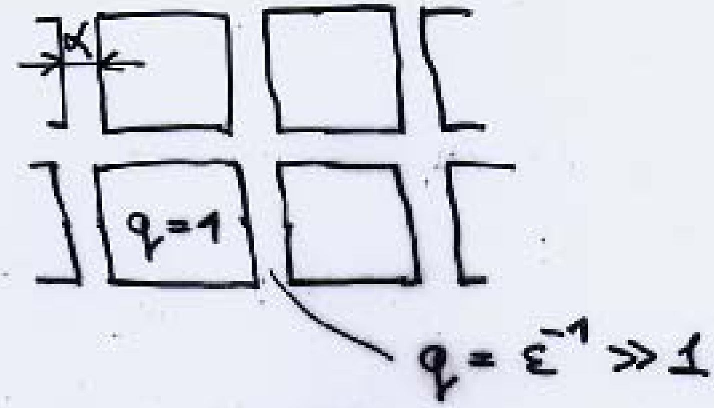
□ $\quad \sigma(L) = \bigcup_n \left[\min_\kappa \lambda_n^\varepsilon(\kappa), \max_\kappa \lambda_n^\varepsilon(\kappa)\right]$

min-max principle

$$\lambda_n^N \le \lambda_n(\kappa) \le \lambda_n^D$$

Figotin, Kuchment 1997

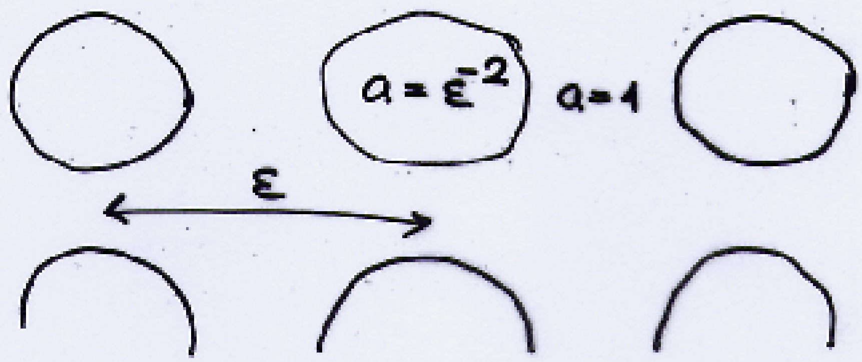
$$-\Delta u = \lambda q u$$



$$\frac{\alpha}{\varepsilon}, \ \frac{\varepsilon}{\alpha^2} \to +\infty \Rightarrow \text{GAPS.}$$

Smyshlyaev, Zhikov 2000-2004.

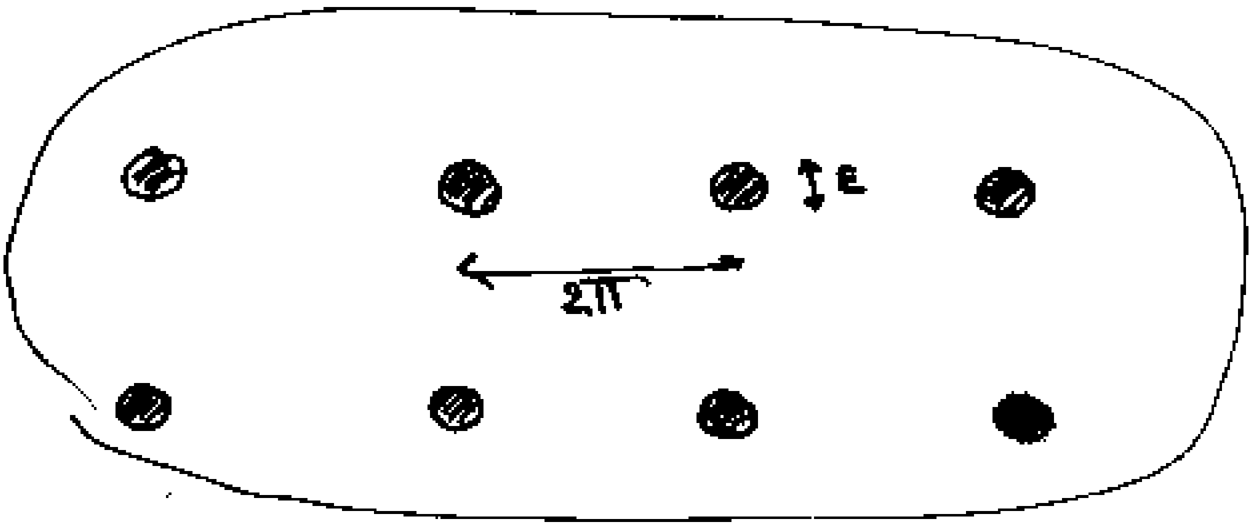
$-\nabla a \nabla u = \lambda u$



Homogenisation. + two-scale convergenc

"double porosity model"

$$(\Delta + \lambda q(x,\varepsilon))\,u(x,\varepsilon) = 0 \quad \text{in } \mathbb{R}^3$$

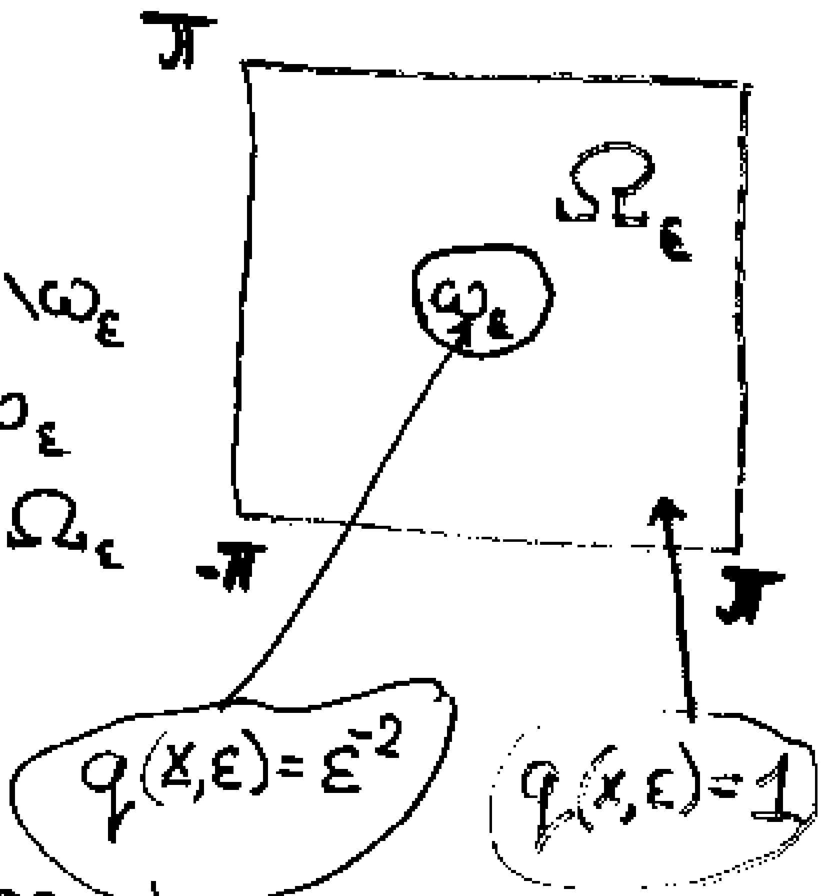


$0 \in \omega \subset \mathbb{R}^3$

$\omega_\varepsilon = \left\{ x \mid \frac{x}{\varepsilon} \in \omega \right\}$

$\Omega = [-\pi, \pi]^3, \ \Omega_\varepsilon = \Omega \setminus \omega_\varepsilon$

$$q(x,\varepsilon) = \begin{cases} \varepsilon^{-2}, & \omega_\varepsilon \\ 1, & \text{in } \Omega_\varepsilon \end{cases}$$

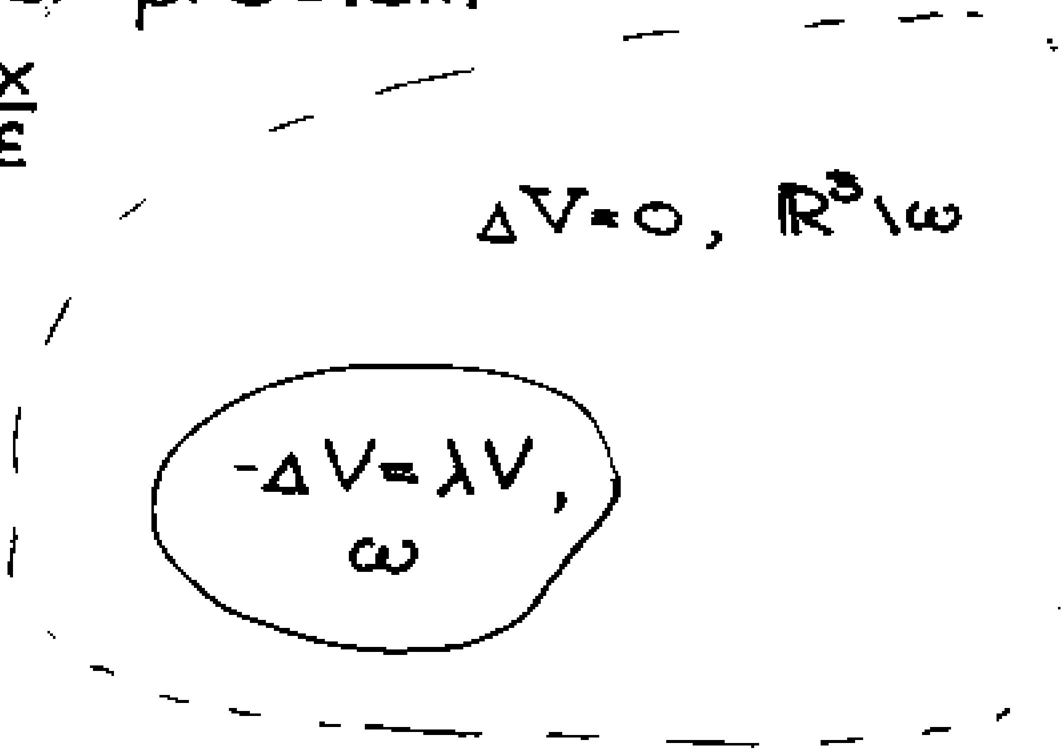


Method of matched asymptotic expansions.

Justification. Maz'ya, Nazarov, Plamenevsky

Inner problem

$x \to \xi = \frac{x}{\varepsilon}$



$$\left(\Delta_\xi + \lambda \rho(\xi)\right) V = 0 , \qquad (1)$$

$$\rho(\xi) = \begin{cases} 1, & \xi \in \omega \\ 0, & \xi \in \mathbb{R}^3 \setminus \omega \end{cases}$$

(1) - spectral problem

$$0 < \Lambda_1 \leq \Lambda_2 \leq \dots \to +\infty$$

$$V_1(\xi), V_2(\xi), \dots$$

$$|V_j(\xi)| \leq \frac{1}{|\xi|}, \quad |\xi| \to +\infty$$

Outer problem

$$-\Delta U = \lambda U \quad , \quad \text{in } [-\pi,\pi]^3$$

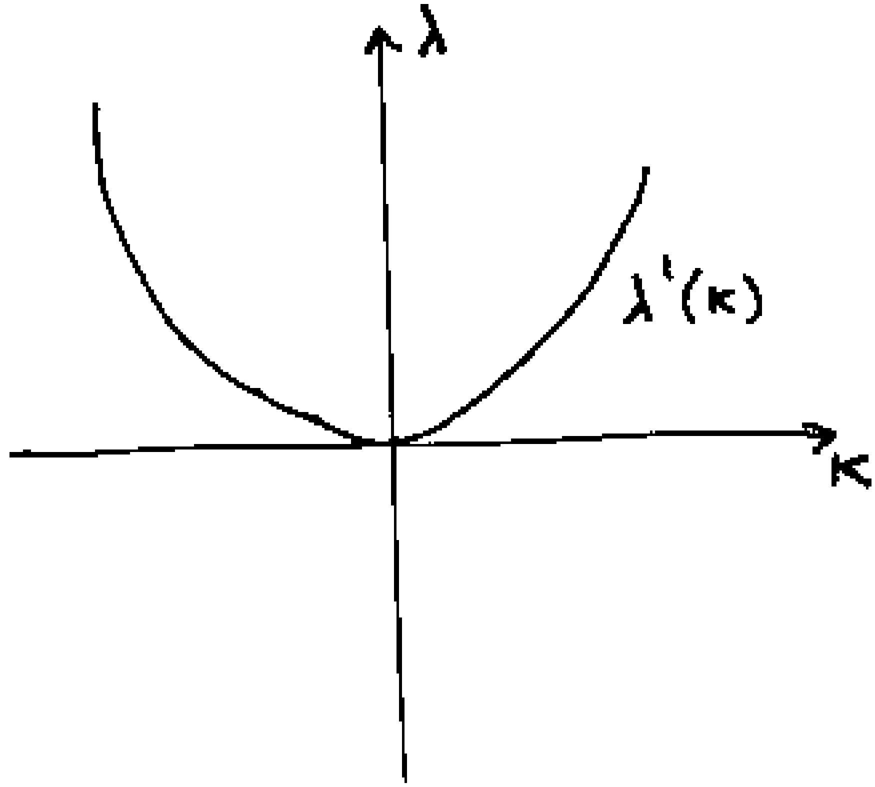
$$+ \text{ q.p. } \quad \underline{k} \in [-\tfrac{1}{2},\tfrac{1}{2})^3$$

Solutions: $\quad \underline{n} \in \mathbb{Z}^3$

$$\lambda_{\underline{n}}(\underline{k}) = |\underline{k}+\underline{n}|^2, \quad U_{\underline{n}}(\underline{k},x) = e^{i(n+k)\cdot x}$$

In particular

$$\lambda^1(\underline{k}) = |\underline{k}|^2, \quad U^1(\underline{k},x) = e^{ik\cdot x}$$



Inner problem

$\Delta V = 0, \quad \mathbb{R}^3 \setminus \omega$

$\Delta V = -\lambda V,$

ω

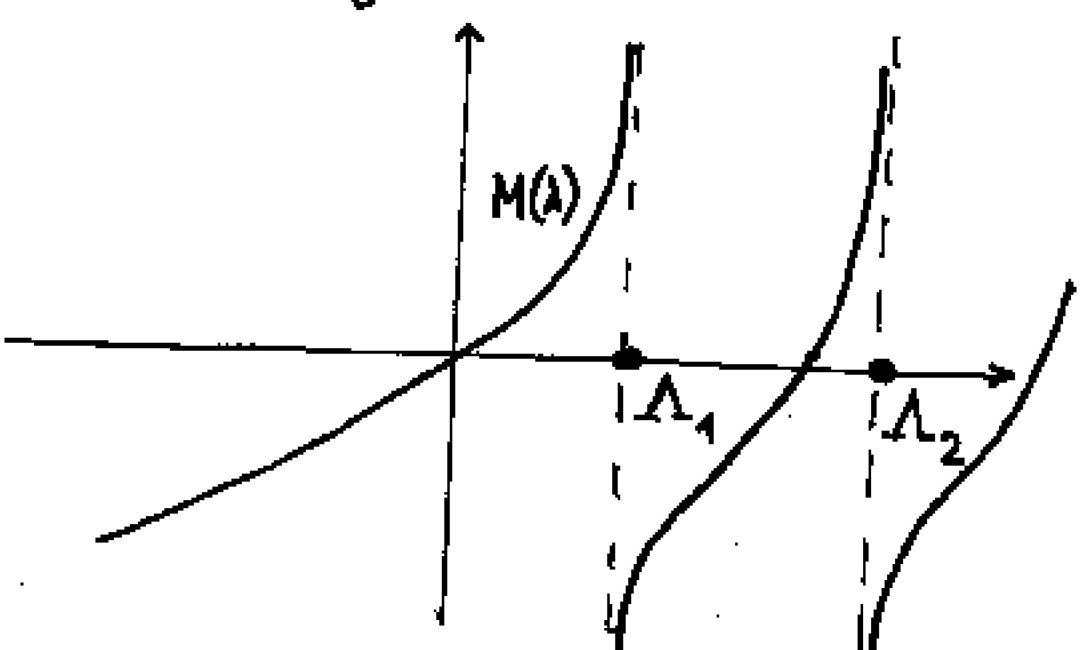
I $\lambda \neq \Lambda_1, \Lambda_2, \dots \Rightarrow \exists$ solution $V(\xi;\lambda)$

$$V(\xi;\lambda) = 1 + M(\lambda)\frac{1}{|\xi|} + \dots \quad , |\xi| \to +\infty$$

Properties :

1. $\dfrac{dM(\lambda)}{d\lambda} > 0$

2. $M(\lambda) = \dfrac{c}{\Lambda_j - \lambda} + \dots \quad , \quad \lambda \to \Lambda_j \, , \; c > 0$



Outer problem

$$-\Delta u = \lambda u, \quad \text{in } [-\pi,\pi]^3 \setminus \underline{0}$$

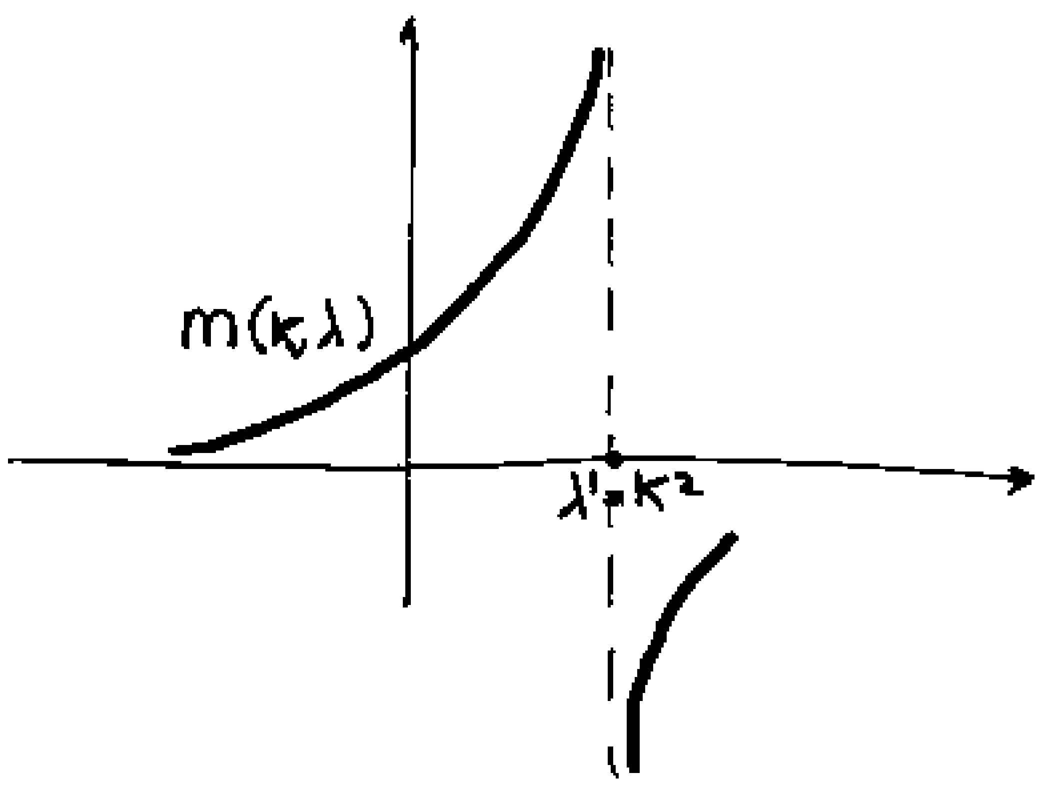
$+$ q.p. $\kappa \in [-\frac{1}{2}, \frac{1}{2})^3$

I $\lambda \neq |n+\kappa|^2$, $n \in \mathbb{Z}^3$ $\Rightarrow \exists$ solution

$$u(x,\kappa,\lambda) = \frac{1}{|x|} + m(\kappa,\lambda) + o(1), \quad |x| \to 0$$

1. $$\frac{d\, m(\kappa,\lambda)}{d\lambda} > 0$$

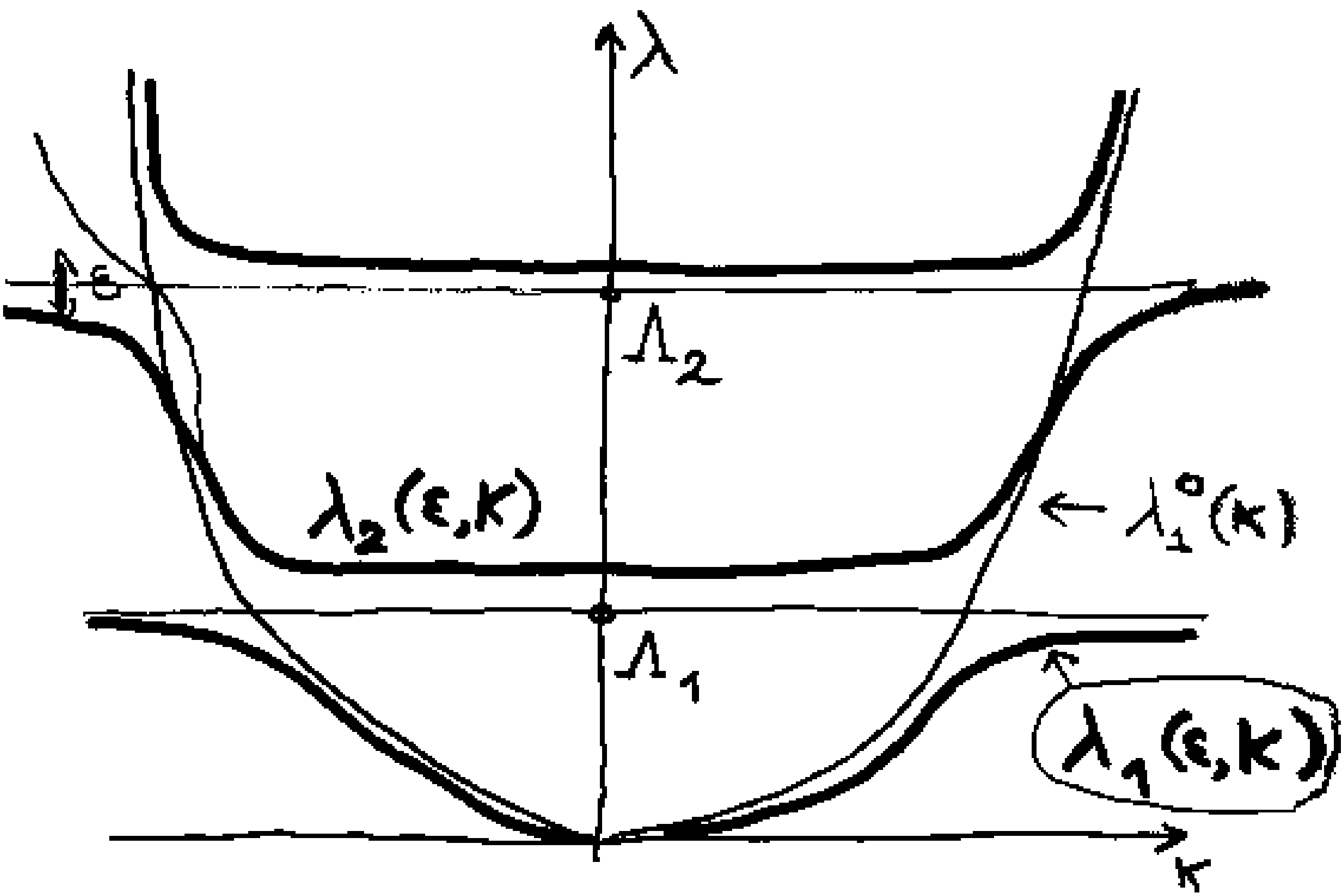
2. $$m(\kappa,\lambda) = \frac{c}{\lambda^1(\kappa)-\lambda} + O(1),$$
$$\lambda \to \lambda^1(\kappa).$$



$$1 = \varepsilon\, m(k,\lambda)\, M(\lambda)$$

Equation with respect to

$$\lambda = \lambda(\varepsilon, k)$$



—— spectrum of outer probl

—— inner problem

—— $\varepsilon \neq 0$

$$L_\varepsilon = \Delta + \lambda\rho(\varepsilon, x), \quad \mathcal{D}(L_\varepsilon) = H^2$$

L.D. Faddeev 1961.

$$L_\varepsilon\big|_{H^2_0} = L, \quad H^2_0 = \{u \in H^2, u(0) = 0\}.$$

$L \subset \mathbb{L}$ - selfadjoint extension

$$\mathcal{D}(\mathbb{L}) = \{u = v + a + b\tfrac{1}{|x|},\ v \in H^2_0,\ b = Pa, \text{ or } a = 0\}.$$

$$\mathbb{L} = \mathbb{L}(p)$$

Asymptotic solution shows.

$$\boxed{P = \varepsilon M(\lambda)}$$

$$L_\varepsilon \cong \mathbb{L}(\varepsilon M(\lambda)) + o(\varepsilon).$$