POINCARÉ–STEKLOV TYPE OPERATORS IN DOMAINS WITH LIPSCHITZ BOUNDARIES[1]

M.S. Agranovich

1. The simplest problems in bounded domains.
2. Exterior problems.
3. Transmission problems.
4. Mixed problems.
5. Crack type problems.
6. Matrix generalizations.

In Section 1, we discuss a typical spectral P-S problem in detail: its setting in Sobolev L_2-spaces, the resolvent set, the smoothness of solutions, and the properties of eigenvalues and eigenfunctions in the cases of a Lipschitz or more smooth boundary. In Sections 2–6, we

---

[1]The investigation was supported by the Russian Foundation for Basic Research.

briefly describe similar problems in other situations.

We use the following notation.

$\Omega = \Omega^+$: a bounded domain in $\mathbb{R}^n$, $n \geq 2$.
The boundary $\Gamma = \partial\Omega$ is Lipschitz.
$\partial_j = \partial/\partial x_j$.
$\nu = \nu(x)$: the exterior unit normal at $x \in \Gamma$ (it exists a.e.).

We consider the equation $Lu = 0$ in Ω, where

$$Lu := \big[-\sum_{j,k} \partial_j a_{jk}(x)\partial_k + \ldots \big] u(x) - \omega^2 u(x).$$

Here $a_{jk}(x) = a_{kj}(x)$ are real, bounded and measurable or, if necessary, more smooth; ... are lower-order terms, in general with variable coefficients;

$\omega = \sigma + i\tau, \quad \tau \geq 0; \quad \sigma \geq 0$ if $\tau = 0$.

The principal symbol: $a(x,\xi) = \sum a_{jk}(x)\xi_j\xi_k$.

We assume that it is elliptic:

$$a(x,\xi) \geq c|\xi|^2, \quad c > 0.$$

The conormal derivative:

$$\partial_{\nu_a}(x) = \sum \nu_j(x) a_{jk}(x) \partial_k \quad (x \in \Gamma).$$

It is the normal derivative ∂_ν for the Helmholtz equation $\Delta u + \omega^2 u = 0$.

The Dirichlet problem (D): $u^+ = g$.
The Neumann problem (N): $\partial_{\nu_a} u^+ = h$.

Assume ω to be such that each of the problems (D) and (N) has a unique solution.

1. The simplest problems in bounded domains. We consider the problem in Ω with boundary condition

$$u^+ = \lambda \partial_{\nu_a} u^+. \tag{1}$$

Here and below, we denote by λ the spectral parameter. We write this equation in the form $\mathbf{N}\varphi = \lambda\varphi$, where $\varphi = \partial_{\nu_a} u^+$ and $\mathbf{N}$ is the operator

$$\mathbf{N} = \mathbf{N}^+ : \ \partial_{\nu_a} u^+ \ \mapsto \ u \ \mapsto u^+; \tag{2}$$

it is called *the Neumann-to-Dirichlet operator*. The inverse operator $\mathbf{D} = \mathbf{D}^+ = \mathbf{N}^{-1}$ is called *the Dirichlet-to-Neumann operator*, and both are called *the Poincaré–Steklov operators*. If Γ and coefficients are C^∞, then $\mathbf{N}$ is an elliptic PSDo of order -1.

To be more precise, we need the Sobolev L_2-*spaces*: $u \in H^{1+s}(\Omega)$, $u^+ \in H^{1/2+s}(\Gamma)$,

$$\partial_{\nu_a} u^+ \in H^{-1/2+s}(\Gamma), \quad |s| < 1/2 \text{ or } \leq 1/2.$$

(If Γ is Lipschitz, then $H^t(\Gamma)$ are defined only for $|t| \leq 1$, but even the trace u^+ of $u \in H^{3/2}(\Omega)$ can lie outside $H^1(\Gamma)$.)

The main form:

$$\Phi_0(u, v) = \int_\Omega \sum a_{jk}(x)\partial_k u(x)\partial_j \overline{v}(x)\, dx.$$

The conormal derivative is actually defined in $H^{-1/2+s}(\Gamma)$ for $u \in H^{1+s}(\Omega)$ by the *variational identity*

$$\Phi_0(u, v) + \int_\Omega (\cdots - \omega^2) u \cdot \overline{v}\, dx = (\partial_{\nu_a} u^+, v^+)_{0,\Gamma}$$

for all v in $H^{1-s}(\Omega)$, $|s| < 1/2$. Here $(\cdot,\cdot)_{0,\Gamma}$ is the usual inner product in $L_2(\Gamma)$ extended to $H^{-1/2+s}(\Gamma) \times H^{1/2-s}(\Gamma)$.

Classical results for $s = 0$. If L is formally self-adjoint, then (D) has a unique solution in $H^1(\Omega)$ for $u^+ \in H^{1/2}(\Gamma)$ and all ω except some real $\omega_j \to +\infty$. The same is true for (N), $\partial_{\nu_a} u^+ \in H^{-1/2}(\Gamma)$, $\omega \neq \widetilde{\omega}_k \to +\infty$. Then $\mathbf{N}$ exists and is invertible as an operator from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. If L is non-self-adjoint, then the exceptional ω can be nonreal, their real parts tend to $+\infty$.

Smoothness: if $\mathbf{D}$ and $\mathbf{N}$ define isomorphisms between $H^{1/2+s}(\Gamma)$ and $H^{-1/2+s}(\Gamma)$ for $s = 0$, then the same is true for $|s| \leq 1/2$.

$\Leftarrow$ Rellich identities, Nečas (1967); Savaré (1998), a new approach, for $|s| < 1/2$.

Now we discuss some *spectral properties of* $\mathbf{N}$. Denote by L_0 the principal part of L. Assume for the beginning that $L = L_0 + \tau^2$, $\tau > 0$.

Remark (B. Pal'tsev, 1996). Then the form

$$\langle \varphi, \psi \rangle_{-1/2,\Gamma} = (\mathbf{N}\varphi, \psi)_{0,\Gamma} \tag{3}$$

is an inner product in $H^{-1/2}(\Gamma)$, the corresponding norm being equivalent to the usual one in this space. Here $(\cdot,\cdot)_{0,\Gamma}$ is extended to $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

In this case, $\mathbf{N}$ is a compact operator with range $H^{1/2}(\Gamma)$, self-adjoint with respect to this inner product, with positive eigenvalues.

In $H^{-1/2+s}(\Gamma)$, introduce the inner products

$$\langle \varphi, \psi \rangle_{-1/2+s,\Gamma} = (\mathbf{N}^{1-2s}\varphi, \psi)_{0,\Gamma}$$

for $-1/2 \leq s \leq 3/2$. In particular, for $s = 1/2$, this is the usual inner product in $L_2(\Gamma)$, and the restriction of $\mathbf{N}$ to this space remains to be a compact self-adjoint operator. The eigenfunctions *belong to* $H^1(\Gamma)$, and it is possible to compose an *orthogonal basis* of them in *all* these spaces with respect to the inner products just indicated.

For other low-order terms in L, we use perturbation argument. The operator $\mathbf{N}$ is either self-adjoint or *a weak perturbation* of a self-adjoint operator. In the last case, the root functions are *complete* in the same spaces. From the root functions, it is possible to compose an Abel-Lidskii basis with parentheses if $n \geq 3$ and even a Riesz basis with parentheses if $n = 2$.

For the eigenvalues $\lambda_j(\mathbf{N})$ numbered in non-increasing order, in general, we have

$$\liminf \lambda_j j^{1/(n-1)} > 0, \quad \limsup \lambda_j j^{1/(n-1)} < \infty.$$

If Γ is C^2 and coefficients $a_{jk}(x)$ are continuous, then

$$\lambda_j(\mathbf{N}) = \alpha j^{-1/(n-1)} + o(j^{-1/(n-1)}), \qquad (4)$$

where α is calculated in terms of $a(x, \xi)$ on Γ (Sandgren, 1955, the variational approach). If the boundary and the coefficients are C^∞, then the remainder term is $O(j^{-2/(n-1)})$ (Hörmander, 1968).

Theorem. *The asymptotic formula* (4) *remains true for C^1 surfaces and continuous coefficients.*

Additional references and more general results will be indicated in Section 4.

The eigenvalue asymptotics is preserved in the non-self-adjoint case.

Simultaneously, we have described the spectral properties of $\mathbf{D} = \mathbf{N}^{-1}$. It is an operator with compact resolvent.

2. Exterior problems. For simplicity, assume that $n \geq 3$. Denote by Ω^- the complement to $\overline{\Omega}$. We assume that L coincides, say, with the Helmholtz operator sufficiently far from the origin and impose the corresponding *radiation condition* on the solutions if $\omega > 0$ and *decay conditions* for other ω. In Ω^-, we assume the solvability and uniqueness for the exterior problems (D) and (N) for given ω; this is true for the Helmholtz equation for all ω.

For the equation $Lu = 0$ in Ω^-, we consider the spectral problem with boundary condition

$$u^- + \lambda \partial_{\nu_a} u^- = 0. \tag{5}$$

The corresponding N-to-D and D-to-N operators are

$$\mathbf{N}^- : \partial_{\nu_a} u^- \mapsto u \,(\text{in } \Omega^-) \mapsto -u^- \tag{6}$$

and $\mathbf{D}^- = (\mathbf{N}^-)^{-1}$. Equation (5) can be written in the form $\mathbf{N}^-\varphi = \lambda\varphi$. The spectral properties of $\mathbf{N}^-$ are similar to those of $\mathbf{N}^+$.

3. Transmission problems. For the equation $Lu = 0$ in $\Omega^+ \cup \Omega^-$, we consider two spectral problems with transmission conditions on Γ. First,

$$u^+ = u^-, \quad u^\pm = \lambda(\partial_{\nu_a} u^+ - \partial_{\nu_a} u^-). \qquad (7)$$

The corresponding N-to-D operator is

$$\mathbf{A} : \partial_{\nu_a} u^+ - \partial_{\nu_a} u^- \ \mapsto u \text{ (in } \Omega^\pm) \ \mapsto u^\pm. \quad (8)$$

Second,

$$\partial_{\nu_a} u^+ = \partial_{\nu_a} u^-, \quad u^+ - u^- = \lambda \partial_{\nu_a} u^\pm. \qquad (9)$$

The corresponding D-to-N operator is

$$\mathbf{C} : u^+ - u^- \ \mapsto u \text{ (in } \Omega^\pm) \ \mapsto \ \partial_{\nu_a} u^\pm. \quad (10)$$

The equations on Γ are

$$\mathbf{A}\varphi = \lambda\varphi \quad \text{and} \quad \psi = \lambda \mathbf{C}\psi, \qquad (11)$$

where $\varphi = \partial_{\nu_a} u^+ - \partial_{\nu_a} u^-$ and $\psi = u^+ - u^-$. The spectral properties of these new operators are similar to those of $\mathbf{N}^+$ and $\mathbf{D}^+$.

Such problems for the Helmholtz equation were proposed by Soviet physicists Katsenelenbaum, Sivov, and Voitovich about 40 years ago. At that time, it was an event to find out that the corresponding operators $\mathbf{N}^\pm$, $\mathbf{A}$ and $\mathbf{C}$ on smooth boundaries are elliptic PSDo with good spectral properties even if they are non-self-adjoint (Agr., 1977).

Transmission conditions arise in diffraction problems on a *half-transparent* surface.

Note that if, for simplicity, the coefficients in L are constant and $E(x)$ is the fundamental solution satisfying our conditions at infinity, then *all these operators are expressed in terms of the surface potentials.* (For the Helmholtz equation in $\mathbb{R}^3$, $E(x) = e^{i\omega|x|}/4\pi|x|$.)

Namely, denote by $\mathcal{A}$ the restriction of the single layer potential to Γ, by $\mathcal{B}$ the direct value of the double layer potential on Γ, and by $\mathcal{C}$ the conormal derivative of the double layer potential on Γ (its values from both sides of Γ are equal; $\mathcal{C}$ is known as the hypersingular operator):

$$\mathcal{A}\varphi(x) = \int_\Gamma E(x-y)\varphi(y)\,dS,$$

$$\mathcal{B}\varphi(x) = \int_\Gamma \partial_{\nu_a(y)} E(x-y)\varphi(y)\,dS,$$

$$\mathcal{C}\varphi(x) = \partial_{\nu_a(x)} \int_\Gamma \partial_{\nu_a(y)} E(x-y)\varphi(y)\,dS.$$

Then

$$\mathbf{A} = \mathcal{A}, \quad \mathbf{N}^\pm = (\mathcal{B} \mp \tfrac{1}{2}I)^{-1}\mathcal{A}, \quad \mathbf{C} = \mathcal{C}.$$

In the *harmonic analysis*, a new approach is developed to surface potentials giving more deep understanding of these operators in Lipschitz domains.

4. Mixed problems. Assume that the Lipschitz boundary Γ is divided into two open parts Γ_1 and Γ_2 by a Lipschitz $(n-2)$-dimensional surface γ. For the equation $Lu = 0$ in Ω, we consider, first, two *non-spectral problems* with boundary conditions

$$\text{(I) } u^+ = g \text{ on } \Gamma_1, \quad \partial_{\nu_a} u^+ = 0 \text{ on } \Gamma_2$$

and

$$\text{(II) } \partial_{\nu_a} u^+ = h \text{ on } \Gamma_1, \quad u^+ = 0 \text{ on } \Gamma_2.$$

Second, we consider two *spectral problems* with boundary conditions

$$\text{(I}'\text{) } u^+ = \lambda \partial_{\nu_a} u^+ \text{ on } \Gamma_1, \ \partial_{\nu_a} u^+ = 0 \text{ on } \Gamma_2$$

and

$$\text{(II}'\text{) } u^+ = \lambda \partial_{\nu_a} u^+ \text{ on } \Gamma_1, \ u^+ = 0 \text{ on } \Gamma_2.$$

The corresponding literature is very rich.

More general mixed problems can be considered, cf. Sandgren (1955), Pal'tsev (1996), but for the problems just indicated we add some details.

As before, we assume that each of problems (D) and (N) in Ω has a unique solution. We introduce the Poincaré–Steklov operators

$$\mathbf{N}_m : \partial_{\nu_a} u^+ (= 0 \text{ on } \Gamma_2) \mapsto u \mapsto u^+|_{\Gamma_1} \quad (12)$$

and

$$\mathbf{D}_m : u^+(= 0 \text{ on } \Gamma_2) \mapsto u \mapsto \partial_{\nu_a} u^+|_{\Gamma_1} \tag{13}$$

($_m$ means *mixed*). Then problems I and II are reduced to the equations

$$\mathbf{N}_m \varphi = g, \quad \mathbf{D}_m \psi = h, \tag{14}$$

and problems I′ and II′ are reduced to the equations

$$\mathbf{N}_m \varphi = \lambda \varphi, \quad \psi = \lambda \mathbf{D}_m \psi. \tag{15}$$

Here $\varphi = \partial_{\nu_a} u^+$ and $\psi = u^+$, but these functions are zero on Γ_2. Actually, we have obtained *the equations on* Γ_1. More precisely:

We now need *more general spaces* ($|s| < 1$, but we will use s only with $|s| \leq 1/2$).

$H^s(\Gamma_1)$: *restrictions* φ *to* Γ_1 *of functions* ψ *from* $H^s(\Gamma)$ with norm $\inf \|\psi\|_{s,\Gamma}$.

$\widetilde{H}^s(\Gamma_1)$: *functions* φ *from* $H^s(\Gamma)$ *supported in* $\overline{\Gamma}_1$ with norm $\|\varphi\|_{s,\Gamma}$.

The following properties of these spaces are well known.

(1) The spaces $H^s(\Gamma_1)$ and $\widetilde{H}^{-s}(\Gamma_1)$ are mutually adjoint with respect to the extension of the usual inner product in $L_2(\Gamma_1)$.

(2) $H^{s_2}(\Gamma_1) \subset H^{s_1}(\Gamma_1)$, $\widetilde{H}^{s_2}(\Gamma_1) \subset \widetilde{H}^{s_1}(\Gamma_1)$ for $s_1 < s_2$; the embeddings are compact, and the left spaces are dense in the right spaces.

(3) The spaces $H^s(\Gamma_1)$ and $\widetilde{H}^s(\Gamma_1)$ can be identified for $|s| < 1/2$.

(4) The operators $\mathbf{N}_m$ and $\mathbf{D}_m$ are bounded in the sense

$\mathbf{N}_m : \widetilde{H}^{-1/2}(\Gamma_1) \mapsto H^{1/2}(\Gamma_1)$,
$\mathbf{D}_m : \widetilde{H}^{1/2}(\Gamma_1) \mapsto H^{-1/2}(\Gamma_1)$.

They are invertible if we exclude some real ω but are not mutually inverse. In view of (2) and (3), our spectral equations $\mathbf{N}_m\varphi = \lambda\varphi, \quad \psi = \lambda\mathbf{D}_m\psi$ make sense, but we have to consider them separately.

Consider problem I. For the beginning, again assume that $L = L_0 + \tau^2, \ \tau > 0$. The invertibility of $\mathbf{N}_m$ is checked by the variational approach. In $\widetilde{H}^{-1/2}(\Gamma_1)$, following Pal'tsev, we introduce the inner product

$$\langle\varphi,\psi\rangle_{-1/2,\Gamma_1}=(\mathbf{N}_m\varphi,\psi)_{0,\Gamma_1} \qquad (16)$$

extending $(\cdot,\cdot)_{0,\Gamma_1}$ to $H^{1/2}(\Gamma_1)\times\widetilde{H}^{-1/2}(\Gamma_1)$.

The operator $\mathbf{N}_m$ in $\widetilde{H}^{-1/2}(\Gamma_1)$ has the range $H^{1/2}(\Gamma_1)$ and is compact and self-adjoint. Using its powers, we introduce the inner products

$$\langle\varphi,\psi\rangle_{-1/2+s,\Gamma_1}=(\mathbf{N}^{1-2s}\varphi,\psi)_{0,\Gamma_1}$$

for $0\le s\le 1$. For $s=1/2$, this is the usual inner product in $L_2(\Gamma_1)$, and the restriction of $\mathbf{N}_m$ to $L_2(\Gamma_1)$ is self-adjoint in this space. The eigenfunctions of $\mathbf{N}_m$ *belong to* $H^{1/2}(\Gamma_1)$, and it is possible to compose an *orthogonal basis* of them with respect to all these inner products.

For more general low-order terms in L, the operator $\mathbf{N}_m$ remains to be self-adjoint or is *a weak perturbation* of a self-adjoint operator. In the main, the picture is the same as in the case of non-mixed problems.

However, no theorem on smoothness is known for mixed problems except for the case of boundaries of the class $C^{1,1}$. For this case, Savaré proved a theorem (1997) similar to his theorem (1998) for non-mixed problems in Lipschitz domains. He noted that, apparently, for piecewise-smooth boundaries the situation is better.

For $L = L_0 + \tau^2$, $\tau > 0$, the $\mathbf{D}_m$ is an operator in $H^{-1/2}(\Gamma_1)$ with domain $\widetilde{H}^{1/2}(\Gamma_1)$ and compact resolvent. It is self-adjoint with respect to the inner product

$$< \varphi, \psi >_{-1/2,\Gamma_1} = (\mathbf{D}_m^{-1}\varphi, \psi)_{0,\Gamma_1}, \qquad (17)$$

here we extend $(\cdot,\cdot)_{0,\Gamma_1}$ to $\widetilde{H}^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)$.

Fortunately, the operator $\mathbf{D}_m^{-1}$ admits a variational definition. Because of this, the picture is essentially the same. In particular, the variational approach to the asymptotics of the eigenvalues works. We now formulate the corresponding results.[2]

[2]The end of this section was somewhat revised at the beginning of October of 2005.

Following Sandgren (1955) and Suslina (1985), we consider more general spectral boundary conditions

$$\rho u^+ = \lambda \partial_{a_\nu} u^+ \text{ on } \Gamma_1, \quad u^+ = 0 \text{ on } \Gamma_2. \quad (18)$$

Here ρ is a bounded measurable function, for the beginning nonnegative. The problem is reduced to the equation

$$\rho \mathbf{D}_m^{-1} \varphi = \lambda \varphi. \quad (19)$$

The formula

$$\lambda_j(\mathbf{D}_m^{-1}) = \alpha_\rho j^{-1/(n-1)} + O(j^{-1/(n-1})) \quad (20)$$

was proved by Sandgren for C^2 boundaries and by Suslina for boundaries piecewise-smooth in the sense of Hestenes -Whitney (C^∞ outside singularities of curvilinear-polyedral type). She considered more general spectral problems and applied a deep variational technique elaborated by M. Birman and M. Solomyak. In both papers, the coefficients are assumed to be continuous.

Returning to (4), note that this formula was proved by Amosov and Agr. (1996) for Lipschitz surfaces that are C^∞ outside a closed set of zero measure and C^∞ coefficients.

Theorem. *If* Γ *is* C^1 *and coefficients* $a_{jk}(x)$ *are continuous, then formula* (20) *is true. Moreover, it remains true if* Γ *is* C^1 *outside a closed set of zero measure.*

In the proof, assuming first that Γ is C^1 and $L = L_0 + \tau^2$, $\tau > 0$, we reduce the general case to the case of a domain under the graph of a positive C^1 function $x_n = \phi(x')$ defined in a ball O on the x'-hyperplane. We transform this domain into a cylinder with $\phi(x') = \text{const}$ and then apply the result by Suslina. If Γ is only C^1 outside a closed set of zero measure, then we additionally use an argument by Suslina.

If ρ changes its sign, then asymptotic formulas for positive and for negative eigenvalues can be written.

It is possible to consider similar problems in Ω^-.

5. Crack type problems. We now consider the case of a non-closed boundary. It was considered in many papers on systems of elasticity theory, where such a boundary has the sense of a crack.

We restrict ourselves by the following problems.

Let Γ_1 be a bounded Lipschitz surface with a Lipschitz boundary γ. Assume that the equation $Lu = 0$ is fulfilled *outside* Γ_1 (with our conditions at infinity), while on Γ_1 the solution is subordinated to some boundary or transmission conditions.

The *non-spectral* conditions on Γ_1:

III. Dirichlet conditions $u^{\pm} = g$.

IV. Neumann conditions $\partial_{\nu_a} u^{\pm} = h$.

The *spectral* conditions on Γ_1:

III′. $u^+ = u^-, \quad u^{\pm} = \lambda[\partial_{\nu_a} u^+ - \partial_{\nu_a} u^-]$.

IV′. $\partial_{\nu_a} u^+ = \partial_{\nu_a} u^-, \quad u^+ - u^- = \lambda \partial_{\nu_a} u^{\pm}$.

Assuming that Γ_1 is a part of a closed Lipschitz surface Γ, we add the following conditions on the remaining part Γ_2 of Γ:

$$u^+ = u^-, \quad \partial_{\nu_a} u^+ = \partial_{\nu_a} u^-,$$

since the equation $Lu = 0$ is fulfilled there. From now on, the situation is very close to that in the case of mixed problems, but the roles of $\mathbf{N}_m$ and $\mathbf{D}_m$ are played by the analogs of the operators $\mathbf{A}$ and $\mathbf{C}$ (i.e., of the single layer potential and hypersingular operator). Namely, we introduce the operators

$$\begin{aligned}\mathbf{A}_{nc} :\ & \partial_{\nu_a} u^+ - \partial_{\nu_a} u^- \ \ (= 0 \ \text{ on } \ \Gamma_2) \\ & \mapsto u \ \ (\text{outside } \ \Gamma_1) \ \mapsto \ u^\pm|_{\Gamma_1}, \\ \mathbf{C}_{nc} :\ & u^+ - u^- \ \ (= 0 \ \text{ on } \ \Gamma_2) \\ & \mapsto u \ \ (\text{outside } \ \Gamma_1) \ \mapsto \ \partial_{\nu_a} u^\pm|_{\Gamma_1}\end{aligned}$$

($_{nc}$ means *nonclosed*). They act as follows:

$$\begin{aligned}\mathbf{A}_{nc} :\ & \widetilde{H}^{-1/2}(\Gamma_1) \mapsto H^{1/2}(\Gamma_1), \\ \mathbf{C}_{nc} :\ & \widetilde{H}^{1/2}(\Gamma_1) \mapsto H^{-1/2}(\Gamma_1).\end{aligned}$$

Problems III and IV are reduced to the equations

$$\mathbf{A}_{nc}\varphi = g \quad \text{and} \quad \mathbf{C}_{nc}\psi = h. \tag{21}$$

Problems III$'$ and IV$'$ are reduced to the equations

$$\mathbf{A}_{nc}\varphi = \lambda\varphi \quad \text{and} \quad \psi = \lambda\mathbf{C}_{nc}\psi. \tag{22}$$

Here the functions $\varphi = \partial_{\nu_a}u^+ - \partial_{\nu_a}u^-$ and $\psi = u^+ - u^-$ are zero on Γ_2, and we have equations on Γ_1.

If $L = L_0 + \tau^2$, $\tau > 0$, then both operators are invertible. Further results are similar to those for our mixed problems. As a byproduct, we obtain an investigation of the Dirichlet and Neumann problems with equal data on (Lipschitz) $\Gamma_1^{\pm}$.

Note that in the case of smooth boundaries, the operators $\mathbf{A}_{nc}$ and $\mathbf{C}_{nc}$, as well as $\mathbf{N}_m$ and $\mathbf{D}_m$, are elliptic PSDo of orders -1 and 1 but *on a manifold* Γ_1 *with boundary.* After initial papers by M. Vishik and G. Eskin (1965-73), such operators were intensively investigated, especially for systems of elasticity theory, by the Wiener-Hopf method by many Georgian and German mathematicians. We mention the papers by R. Duduchava–D. Natroshvili–E. Shargorodskii (1990) and Duduchava–Wendland (1991).

This approach leads to a *construction of a parametrix*. Note that it is possible to give a more transparent construction for it than in the general theory of Vishik and Eskin. However, some smoothness is necessary.

The variational approach, which we use, gives the results for Lipschitz boundaries. A simplified version of this part or the talk is in Agr., 2006 (in print).

6. Generalizations to systems. Now we consider a system $Lu = 0$, in which the coefficients are $m \times m$ matrices. The main examples are the systems of the elasticity theory.

The conormal derivative is a matrix operator. The principal symbol is assumed to be real, symmetric and strongly elliptic:

$$a(x, \xi) = \sum \xi_j \xi_k a_{jk}(x) \geq c|\xi|^2 E, \quad c > 0,$$

where E is the $m \times m$ unit matrix. It follows that the variational approach can be applied to the Dirichlet problem.

To apply it to the Neumann problem, say, in Ω, we assume that the principal form $\Phi_0(u, v)$ is *coercive on* $H^1(\Omega)$. In Agr., 2002, some sufficient conditions are indicated suggested by the Korn inequality.

To formulate *the radiation conditions*, we assume that the principal symbol does not depend on x in a neighborhood of infinity, impose some conditions on its eigenvalues and then follow B. Vainberg, 1996. See Agr., 2002.

There is a book by McLean, 2000, devoted to the variational approach for systems (including the mixed problems) and to equations on the boundary.

The Rellich identities work in general only in one direction: if the Dirichlet data are smooth, then the Neumann data are smooth. (For the Lamé system, they work in both directions.) However, *Savare's theorem* on smoothness (1998) is extended to systems. See Agr., 2002.

In the main, the spectral results are the same as for the scalar equation.