Localisation in the parabolic Anderson and Bouchaud trap models

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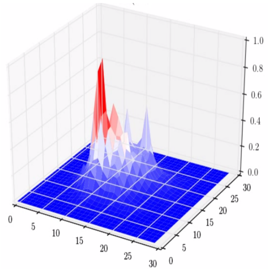
April, 2014



A simulation of the parabolic Anderson model

The parabolic Anderson model

The parabolic Anderson model is the continuous-time branching random walk on $\mathbb{Z}^d$ defined by:

1. **Initialisation**:

   (a) **Initial state**: A single particle at the origin;

   (b) **Random environment**: A random field on $\mathbb{Z}^d$

   $$\xi := \{\xi(z)\}_{z \in \mathbb{Z}^d}$$

   consisting of i.i.d. strictly-positive RVs known as the **random potential field**.

2. **Dynamics**:

   (a) **CTSRW**: All particles undertake independent continuous-time simple random walks on $\mathbb{Z}^d$;

   (b) **Branching**: A particle at size z branches at rate $\xi(z)$.

We are interested in the **mass function** of the model:

$$u(t, z) := \mathbb{E}_{\mathsf{RW}} \left[\text{“}\# \text{ of particles at site } z \text{ at time } t\text{”}\right]$$

where $\mathbb{E}_{\mathsf{RW}}$ denotes that the expectation is taken over realisations of the branching random walk in the **fixed** random environment.

Clearly, $u(t, z)$ is a random variable depending on the particular realisation of ξ. In the language of statistical mechanics, this is the **quenched** (as opposed to the **annealed**) mass function.

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Since we're taking an expectation, we can simplify things by weighting the trajectories of a single CTSRW:

$$u(t, z) = \mathbb{E}_{\mathrm{RW}} \left[\exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}\{X_t = z\} \right]$$

where $\{X_t\}_{t \geq 0}$ is a continuous-time simple random walk on $\mathbb{Z}^d$.

PDE formulation

Via the Feynman-Kac formula, we can alternatively consider $u(t,z)$ as the the solution of the following Cauchy problem:

$$\frac{\partial u(t,z)}{\partial t} = (\Delta + \xi)u(t,z) \qquad\qquad u(0,z) = \mathbb{1}_{\{0\}}(z)$$

where Δ is the Laplacian on $\mathbb{Z}^d$ and $\mathbb{1}_{\{0\}}$ is the indicator function of the origin.

Under mild moment conditions on $\xi(\cdot)$, a unique solution $u(t,z)$ exists almost surely [Gärtner and Molchanov, 1990].

Link to quantum physics

The PAM has its origins in the statistical physics literature, where it was introduced by P.W. Anderson in the 1960s to model the behaviour of electrons inside a semiconductor.

Recall the time-independent Schrödinger equation

$$i\hbar\frac{\partial u(t,z)}{\partial t} = \left(\frac{-\hbar^2}{2m}\Delta + \xi\right)u(t,z)\,.$$

We call the operator $\Delta + \xi$ a random Schrödinger operator.

The Bouchaud trap model

The Bouchaud trap model is the CTRW on $\mathbb{Z}^d$ defined by:

1. **Initialisation**:
   (a) **Initial state**: A single particle at the origin;
   (b) **Random environment**: A random field on $\mathbb{Z}^d$

$$\sigma := \{\sigma(z)\}_{z \in \mathbb{Z}^d}$$

   consisting of i.i.d. strictly-positive RVs known as the **random trapping landscape**.

2. **Dynamics**: The particle undertakes a CTRW on $\mathbb{Z}^d$ with jump rates

$$\tau(z \to y) := \begin{cases} \frac{1}{2d} \frac{1}{\sigma(z)} & \text{if } y \sim z \\ 0 & \text{else} \end{cases} .$$

Mass function

We are interested in the **mass function** of the model:

$$u(t, z) := \mathbb{P}_{\mathrm{RW}}\left[\text{“the particle is at site } z \text{ at time } t\text{”}\right]$$

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where $\mathbb{P}_{\mathrm{RW}}$ denotes the probability with respect to the random walk in the **fixed** random environment.

As in the PAM, there is a PDE formulation for $u(t,z)$:

$$\frac{\partial u(t,z)}{\partial t} = \Delta \sigma^{-1} u(t,z) \qquad\qquad u(0,z) = \mathbb{1}_{\{0\}}(z)\,.$$

The BTM also has its origins in the statistical physics literature, where it introduced by Bouchaud in the 90s as a toy model for the long-term behaviour of spin-glasses.

Intermittency and localisation

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More precisely, we say that the PAM and the BTM localises if there exists a (random) localisation set Γ_t such that $|\Gamma_t| = t^{o(1)}$ and

$$\frac{\sum_{z \in \Gamma_t} u(t,z)}{U(t)} \to 1 \quad \text{in probability} \tag{1}$$

where $U(t) := \sum_{z \in \mathbb{Z}^d} u(t,z)$ is the total mass of the process.

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We describe the localisation as **complete** if $|\Gamma_t|$ can be chosen in equation (1) such that $|\Gamma_t| = 1$.

Almost sure localisation is the stronger statement where the convergence in equation (1) is almost sure.

Localisation strength

Broadly speaking, the strength of localisation in the PAM and BTM depends on:

1. the asymptotic **rate of decay**; and
2. the **regularity**,

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Broadly speaking, the strength of localisation in the PAM and BTM depends on:

1. the asymptotic **rate of decay**; and
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It is convenient to characterise $\xi(\cdot)$ and $\sigma(\cdot)$ by their **exponential tail decay rate** functions

$$f_\xi(x) := -\log(\mathbb{P}(\xi(\cdot) > x)) \quad \text{and} \quad f_\sigma(x) := -\log(\mathbb{P}(\sigma(\cdot) > x)) \,.$$

For simplicity, we will assume maximum regularity for the tails.

Localisation strength

We call the connected components of the localisation set Γ_t the **localisation islands**.

It is natural to characterise the strength of localisation by studying Γ_t along two dimensions:

1. The **number** of localisation islands ;
2. The **size** of each island.

Localisation in the PAM: Known results

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Tail decay	$\log f_\xi(x)$	**No. loc. isl.**	**Size loc. isl.**
(1) (Almost)-bounded	$\gg x$	(Growing?)	Growing
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(3) Sub-double exp.		$\ll x$	(Single)	Single
Stretched.-D.E.	$\beta < 1$	x^β	(Single)	Single
Weibull	$\gamma \geq 2$	$\gamma \log x$	(Single)	Single
	$\gamma < 2$	$\gamma \log x$	Single	Single
Pareto		$\log \log x$	Single	Single

Results on the size of the islands is due to [Gärtner, König and Molchanov, 2007]. The Pareto case was done in [van der Hofstad, Mörters and Sidorova, 2008]. The sub-normal Weibull case ($\gamma < 2$) was done in [Sidorova and Twarowski, 2012].

Our results

Theorem

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2. A limit theorem for the localisation distance;
3. A limit theorem describing the potential field near the localisation site;
4. That the localisation site exhibits *ageing* (i.e. the time between successive relocalisations grows linearly).

An aside: Are our results physically meaningful?

Our exponential decay result implies that

$$\frac{u(t,z)}{U(t)} \approx e^{\frac{-|z-Z_t^{(1)}|}{\gamma}\log\log t}$$

where $Z_t^{(1)}$ denotes the localisation site, and so

$$\sum_{z \neq Z_t^{(1)}} \frac{u(t,z)}{U(t)} \approx 4d\, e^{-\frac{1}{\gamma}\log\log t}$$

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So to ensure that

$$\frac{u(t,Z_t^{(1)})}{U(t)} > \frac{1}{2}$$

we need

$$t \approx \exp\left(\exp\left(\gamma \log(4d)\right)\right) .$$

In the case $d = 3$ with normal tails ($\gamma = 2$), this requires $t \approx 10^{62}$, older than the current age of the universe in Planck time. If $\gamma = 3$, we would have to wait until the eventual heat death of the universe to see localisation.

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1. The **benefit** of being at a **region of high potential**, i.e. where lots of high potential sites are clustered;
2. The **cost of diffusing** too far too quickly.

So we expect that

$$Z_t^{(1)} = \text{argmax}_{z \in \mathbb{Z}^d} \Psi_t(z)$$

where

$$\begin{aligned}\Psi_t(z) &= \text{"benefit of being near site } z\text{"} \\ &\qquad - \text{"cost of } z \text{ being too far from the origin"} \\ &= f_t\left(\{\xi(\cdot)\}_{\text{near } z}\right) \quad - \quad g_t(|z|) \quad \text{for some } f_t, g_t\end{aligned}$$

Describing the localisation site

The correct functional is

$$\Psi_t^{(\rho)}(z) := \tilde{\lambda}_t^{(\rho)}(z) \quad - \quad \frac{|z|}{\gamma t} \log \log t$$

where $\tilde{\lambda}_t^{(\rho)}(z)$ is the principle eigenvalue of the Hamiltonian $\mathcal{H} = \Delta + \xi$ restricted to a ρ-ball around z, for a certain constant $\rho = \lfloor (\gamma - 1)^+ / 2 \rfloor$ that depends on the Weibull parameter γ.

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The constant ρ is the **radius of localisation**, i.e. the radius at which the potential field of neighbouring sites will influence localisation. Note that $\rho = 0$ if $\gamma < 3$, and so in that case localisation depends only on ξ as a scalar field.

Outline of proof

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Step 2: Consider the spectral representation of the solution:

$$u_{V_t}(t,z) = \sum_{i=1}^{|V_t|} e^{t\lambda_{t,i}} \varphi_{t,i}(0)\, \varphi_{t,i}(z) = \sum_{i=1}^{|V_t|} e^{t\Psi_{t,i}}\, \varphi_{t,i}(z)$$

where

$$\Psi_{t,i} := \lambda_{t,i} + \frac{\log|\varphi_{t,i}(0)|}{t}.$$

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where

$$\Psi_{t,i} := \lambda_{t,i} + \frac{\log|\varphi_{t,i}(0)|}{t}\,.$$

If we can establish a gap in the maximisers of $\Psi_{t,i}$, larger than order $1/t$, then the spectral representation will be asymptotically dominated by just one eigenfunction. Complete localisation and exponential decay is then inherited from the exponential decay of the dominating eigenfunction.

Outline of proof

Step 3: Approximate $\Psi_{t,i}$ with $\Psi_t^{(\rho)}(z_i)$ up to a certain error, where $z_i = \arg\max_{z \in V_t}\{\varphi_{t,i}(z)\}$.

To do this we show that

1. $\lambda_{t,i} \approx \tilde{\lambda}_t^{(\rho)}(z_i)$ 'eigenvalues lack resonance'
2. $\varphi_{t,i}(0) \approx (\log t)^{-|z_i|/\gamma}$ 'exponential decay of eigenfunctions'

both up to a certain error.

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Step 4: Establish that the 'gap' in the maximisers of the $\Psi_t^{(\rho)}(z)$ exceeds both the order $1/t$ and the order of the error in step 3. For this step we use point process techniques.

Step 4: The point process approach

We rescale the penalisation functional

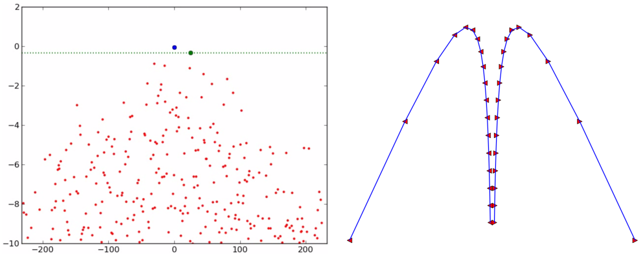
$$Y_{t,z} := \frac{\Psi_t^{(\rho)}(z) - A_{r_t}}{d_{r_t}} \quad \text{and} \quad \mathcal{M}_t := \sum_{z \in V_t} \mathbb{1}_{(z r_t^{-1}, Y_{t,z})}$$

using the scales

1. $A_t \sim \max_{z \in V_t} \tilde{\lambda}^{(\rho)}(z)$, for the extremes of the local eigenvalues;
2. $d_t := \frac{dA_t}{dt}$, for the 'gaps' in the extremes of the local eigenvalues;
3. $r_t := \frac{d_t}{\log A_t}$, for the localisation distance.

Step 4: The point process approach

We show that, on these scales, the set $\mathcal{M}_t$ converges to a point process in the limit. This establishes that the gap between the maximises of $\Psi_t^{(\rho)}(z)$ is of the order d_t.



The point set $\mathcal{M}_t$, and the trajectories of points in $\mathcal{M}_t$ over time.

Localisation in the BTM: Known results

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In one dimension, there appears to be three distinct regimes of localisation:

Tail decay	$f_\sigma(x)$	**Localisation**
(1) Light-tail (i.e. finite mean)	$\gg \log x$	No intermittency
(2) Heavy-tail/Pareto, $c \in (0, 1)$	$\sim c \log x$	Intermittency, but no localisation
(3) Super-heavy-tail	$\ll \log x$	??

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(3) Super-heavy-tail	$\ll \log x$	??
log-Weibull $\gamma < 1$	$(\log x)^\gamma$	
log-Pareto	$\log \log x$	

Results in the light-tail/Pareto cases due to [Fontes, Isopi and Newman, 1999/2012].

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We describe the localisation site explicitly, and prove a number of related results, including:

1. A limit theorem for the localisation distances;
2. That the mass function on the localisation sites is distributed as Dirichlet$(1, 1)$ in the limit (i.e. the proportion at each site is uniformly distributed).

Describing the localisation site

The two localisation sites $Z_t^{(1)}$ and $Z_t^{(2)}$ take the form of the first traps on both the **positive** and **negative** half-line whose depth exceeds a certain level l_t, which we define as the unique solution to the equation

$$f_\sigma(l_t) + \log l_t = \log t\,.$$

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$$f_\sigma(l_t) + \log l_t = \log t \,.$$

The level l_t is chosen to be:

1. Small enough so that the particle has a strong chance of hitting Γ_t before time t; but
2. Large enough such that, if the particle hits $z \in \Gamma_t$ before time t, it has a strong chance of still being at z at time t.

Outline of proof

The underlying fact behind the result is:

Proposition

In an i.i.d. sequence of super-heavy-tailed RVs, the running maximum asymptotically dominates the cumulative sum.

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Step 1: Bound in probability the time until the particle hits Γ_t.

Use Ray-Knight type results to bound this time above by the sum of the trap depths between the two sites in Γ_t multiplied by the distance to Γ_t. By the Proposition, this time is overwhelmingly likely to be less than t.

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Step 3: Condition on the particle hitting $z \in \Gamma_t$ by time t, and place periodic boundary conditions on the box. Since

$$(\Delta\sigma^{-1})u = 0 \implies \Delta(\sigma^{-1}u) = 0\,,$$

the equilibrium distribution in the box is proportionate to the trapping landscape, which, by the Proposition, is dominated by the trap at z. Since we converge monotonically downwards to equilibrium, the mass is also dominated by z.

Open questions

Lots of interesting questions remain:

1. In the PAM, a major open question is whether the PAM **always** localises on just one island.
2. In the BAM, the nature of the transition from two-site localisation (super-heavy tails) to delocalisation (heavy-tails) is unclear. Are there intermediate phases?

References

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2. S. Muirhead, *Two-site localisation in the Bouchaud trap model with super-heavy-tailed traps* (2014), arXiv:1402.4983