Cell Centered Finite Volume Schemes for Multiphase Flow Applications

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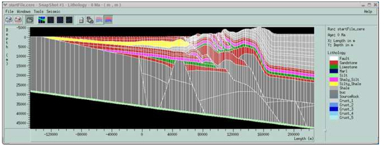
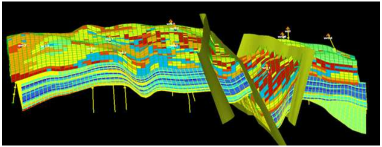
Applications

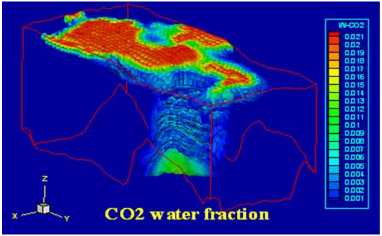
Basin Modeling



Reservoir simulation



CO_2 geological storage



Motivations of cell centered schemes for compositional multiphase Darcy flow applications

- Explicit linear fluxes
- Pressure, Saturations, Compositions all defined at the cell centers
- Existing Efficient Preconditioners like CPR-AMG

- But cell centered VF schemes are non symmetric on general meshes
  - Possible lack of robustness due to mesh and permeability dependent coercivity

Model problem

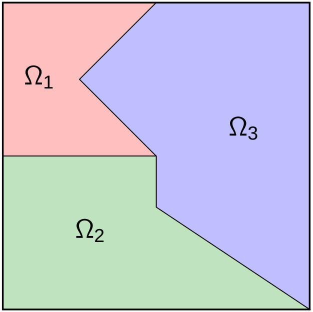
- Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain
- For $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} -\mathrm{div}(\nu \nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

- Let $a(u, v) = \int_\Omega \nu \nabla u \cdot \nabla v$. The weak formulation reads

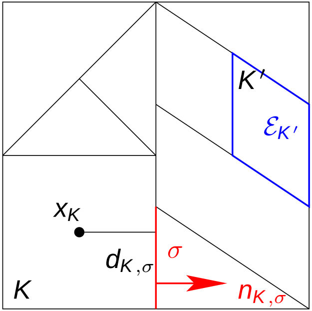
$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = \int_\Omega fv \text{ for all } v \in H_0^1(\Omega) \qquad (\Pi)$$

Model problem



- Let $\{\Omega_i\}_{1 \leq i \leq N_\Omega}$ be a partition of Ω into bounded polygonal sub-domains
- $\nu|_{\Omega_i}$ smooth and $\nu(x)$ is s.p.d. for a.e. $x \in \Omega$

Polyhedral admissible meshes



$\mathcal{T}_h$: set of cells K
$\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$: set of inner and boundary faces σ
m_σ: surface of the face σ
m_K: volume of the cell K

Discrete function space V_h

- V_h: space of piecewise constant functions on $\mathcal{T}_h$

$$v_h(x) = v_K \text{ for all } x \in K$$

- Equip V_h with the following discrete H^1_0 norm:

$$\forall v_h \in V_h, \quad \|v_h\|_{V_h} = \left(\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} |\gamma_\sigma(v_h) - v_K|^2 \right)^{1/2}$$

- using the following trace reconstruction at the faces σ

$$\begin{cases} \gamma_\sigma(v_h) = \text{ if } \sigma = \frac{v_K d_{L,\sigma} + v_L d_{K,\sigma}}{d_{L,\sigma} + d_{K,\sigma}} \text{ if } \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i, \\ \gamma_\sigma(v_h) = 0 \text{ if } \sigma \in \mathcal{E}_h^b. \end{cases}$$

Finite Volume Scheme

- Let $F_{K,\sigma}(u_h)$ denote a conservative linear approximation of $\int_\sigma \nu \nabla u \cdot n_{K,\sigma}$

$$\text{conservativity: } F_{K,\sigma}(u_h) + F_{L,\sigma}(u_h) = 0, \quad \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i.$$

- The finite volume scheme reads

$$\text{find } u_h \in V_h \text{ such that}$$

$$-\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_K f \quad \text{for all } K \in \mathcal{T}_h.$$

Discrete variational formulation

- For all $u_h, v_h \in V_h$, let

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h)(\gamma_\sigma(v_h) - v_K)$$

$$= \sum_{\sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h) v_K$$

- The finite volume scheme is equivalent to:

find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h.$$

A symmetric scheme: using a discrete variational framework [Eymard and Herbin, 2007]

- Discrete variational formulation

$$a_h(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h.$$

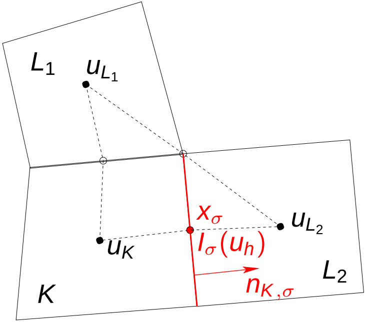
- with a_h based on a cellwise constant gradient and stabilized by residuals

$$\begin{aligned} a(u_h, v_h) = & \sum_{K\in\mathcal{T}_h} m_K \nu_K (\nabla_h u_h)_K \cdot (\nabla_h v_h)_K \\ & + \sum_{K\in\mathcal{T}_h} \eta_K \sum_{\sigma\in\mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} R_{K,\sigma}(u_h) R_{K,\sigma}(v_h) \end{aligned}$$

Discrete gradient reconstruction

- The cellwise constant gradient is obtained via the Green formula and trace reconstruction

$$(\nabla_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (I_\sigma(v_h) - v_K) n_{K,\sigma}$$



- Residuals:

$$R_{K,\sigma}(v_h) = I_\sigma(v_h) - v_K - (\nabla_h v_h)_K \cdot (x_\sigma - x_K)$$

Convergence

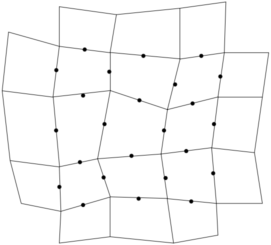
- Coercivity of the bilinear form a_h
- Consistency of the discrete gradient
- Weak convergence property of the discrete gradient (Rellich Theorem)

Conservative Fluxes

- Fluxes are derived from the bilinear form:

$$a_h(u_h, v_h) = \sum_{KL} F_{K,L}(u_h)(v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h) v_K$$

- Fluxes $F_{K,L}(u_h)$ are between cells K and L not necessarily sharing a face
- Stencil of the scheme: example for topologicaly cartesian grids
  - 21 cells in 2D, 81 cells in 3D

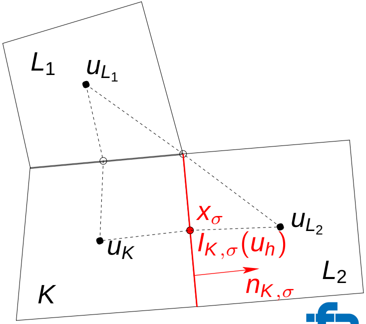


The GradCell Scheme: non symmetric scheme [Agélas et al., 2008]

- Use two cellwise constant gradients: a consistent gradient and a weak gradient

$$(\nabla_h v_h)_K = \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma (I_{K,\sigma}(v_h) - v_K) n_{K,\sigma}$$

$$(\widetilde{\nabla}_h v_h)_K = \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma (\gamma_\sigma(v_h) - v_K) n_{K,\sigma}$$

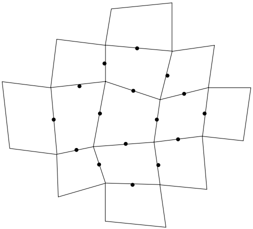


The GradCell scheme: Conservative Fluxes

- Fluxes are derived from the bilinear form:

$$a_h(u_h, v_h) = \sum_{\sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h) v_K$$

- Fluxes $F_{K,\sigma}(u_h)$ only between cells sharing a face
- The stencil of the scheme becomes much sparser: neighbours of the neighbours
- Example for topologically cartesian grids
  - 13 cells in 2D, 21 cells in 3D



Convergence

- Consistency of the consistent gradient
- Weak convergence property of the weak gradient
- Coercivity is mesh and diffusion tensor dependent

How to acheive both symmetry and sparse stencil: SUSHI (Scheme Using Stabilization and Harmonic Interfaces) [Agelas et al., 2009]

Combine

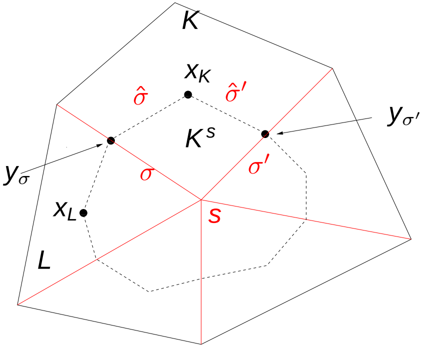
- O scheme ideas: subcell gradients $(\nabla u_h)_K^s$ and subfaces unknowns u_σ^s
- Weak and consistent gradient definition
- Two point harmonic interpolation at the faces

To obtain

- Symmetric unconditionally coercive scheme
- Sparse stencil: 9 points in 2D and 27 points in 3D on topologically Cartesian meshes

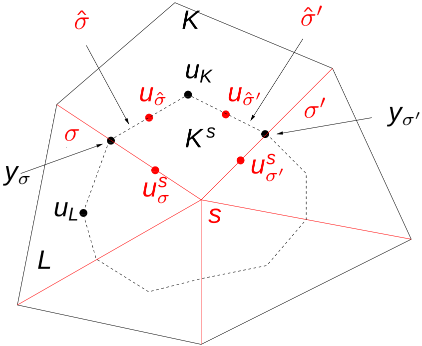
Subcells K_s around a vertex s

- Choose the points $y_\sigma \in \sigma$ for all faces (harmonic points)
-
$$K_s = \left(x_K, y_\sigma, s, y_{\sigma'}, x_K \right)$$



Discrete gradient on a subcell K_s

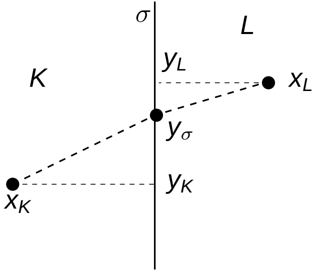
$$(\nabla_h u)_{K_s} = \frac{1}{m_{K_s}}\Big(m_\sigma^s (u_\sigma^s - u_K) n_{K,\sigma} + m_{\sigma'}^s (u_{\sigma'}^s - u_K) n_{K,\sigma'} + m_{\hat\sigma} (u_{\hat\sigma} - u_K) n_{K_s,\hat\sigma} + m_{\hat\sigma'} (u_{\hat\sigma'} - u_K) n_{K_s,\hat\sigma'} \Big)$$



Harmonic point y_σ and harmonic interpolation

There exists a point $y_\sigma \in \sigma$ and a coefficient α with linear two point interpolation exact on piecewise linear functions with normal flux and potential continuity

$$u(y_\sigma) = \alpha\, u(x_K) + (1 - \alpha)\, u(x_L)$$



Hybrid variational formulation

$$\mathcal{H}_h = \left\{ (u_K)_{K\in\mathcal{T}_h}, (u_\sigma^s)_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}_h}, \text{ s. t. } u_\sigma^s = 0 \text{ for all } \sigma \in \mathcal{E}_h^b \right\}.$$

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h}\sum_{s\in\mathcal{V}_K}\left(m_{K_s}\nu_K(\nabla_h u)_K^s\cdot(\nabla_h v)_K^s + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_{K_s}}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u_h) R_{K,\sigma}^s(v_h)\right)$$

Finite volume scheme: find $u_h \in \mathcal{H}_h$ such that

$$a_h(u_h,v) = \sum_{K\in\mathcal{T}_h} v_K \int_K f(x)dx \quad \text{for all } v \in \mathcal{H}_h.$$

Hybrid Finite Volume scheme

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \sum_{s\in\mathcal{V}_K} \sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} \sum_{\sigma'\in\mathcal{E}_s\cap\mathcal{E}_K} (T_K^s)_{\sigma,\sigma'}(u_{\sigma'}^s - u_K)(v_\sigma^s - v_K),$$

Let us define the following subfluxes:

$$F_{K,\sigma}^s(u) = \sum_{\sigma'\in\mathcal{E}_s\cap\mathcal{E}_K} (T_K^s)_{\sigma,\sigma'}(u_{\sigma'}^s - u_K),$$

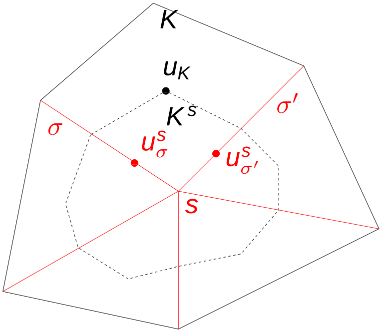
Hybrid Finite volume scheme:

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_K} \Big(\sum_{s\in\mathcal{V}_\sigma} F_{K,\sigma}^s(u_h) \Big) = \int_K f(x)dx & \text{for all } K \in \mathcal{T}_h, \\ F_{K,\sigma}^s(u_h) = -F_{L,\sigma}^s(u_h) & \text{for all } s \in \mathcal{V}_\sigma, \sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h^i. \end{cases}$$

Comparison with the O scheme

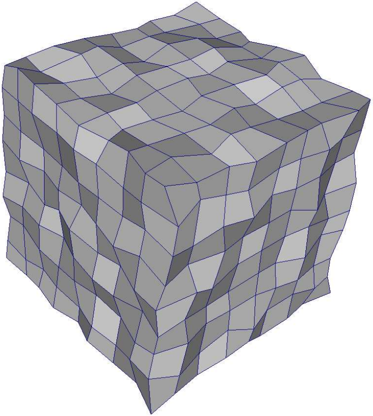
$$(\nabla_h u)_K^s = \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} (u_\sigma^s - u_K)\, g_{K,\sigma}^s$$

$$(\widetilde{\nabla}_h u)_K^s = \frac{1}{\mathrm{m}_{K_s}} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s (u_\sigma^s - u_K) n_{K,\sigma}.$$



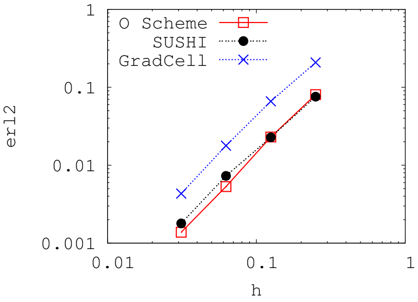
$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \sum_{s\in\mathcal{V}_K} \mathrm{m}_{K_s} (\nabla_h u)_{K_s} \cdot \nu_K (\widetilde{\nabla}_h v)_{K_s}$$

Randomly distorted Cartesian meshes

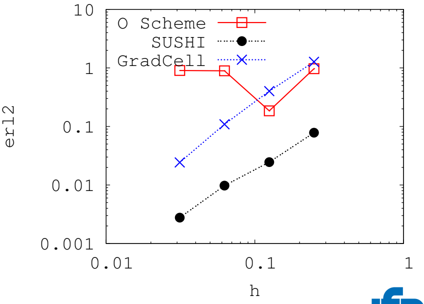


Convergence for randomly distorted Cartesian meshes and smooth solution

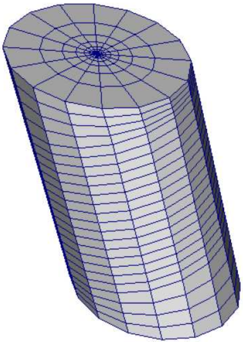
Anisotropy $\nu = \text{diag}\left(1, 1, 100\right)$

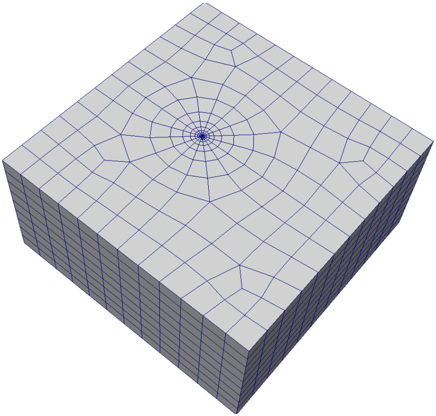


Anisotropy $\nu = \text{diag}\left(1, 1, 1000\right)$



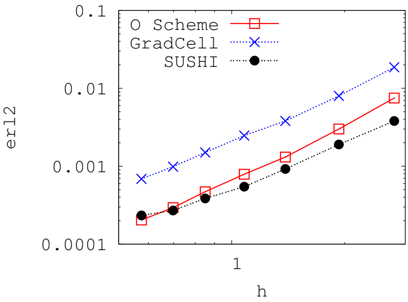
Near well hexahedral meshes



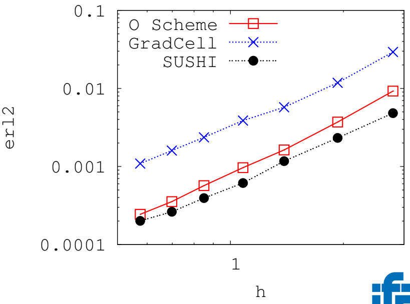


Convergence for near well hexahedral meshes and deviated well analytical solution

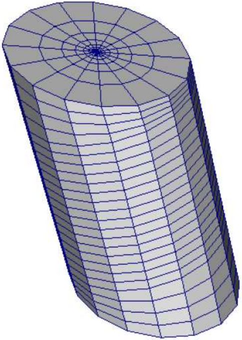
$$\nu = \mathrm{diag}\left(1, 1, 0.2\right)$$

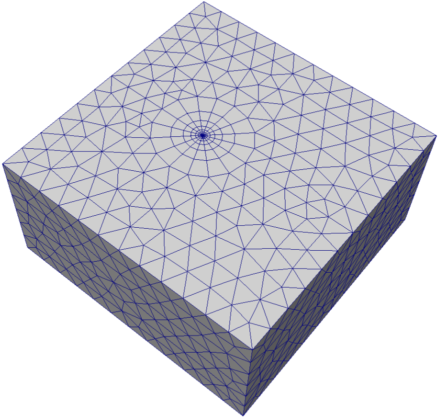


$$\nu = \mathrm{diag}\left(1, 1, 0.05\right)$$

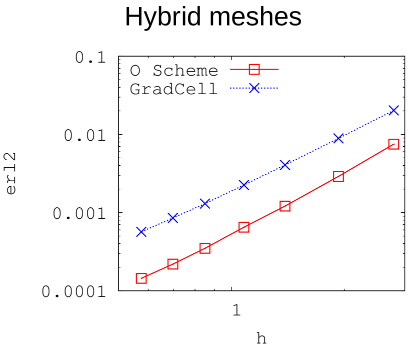


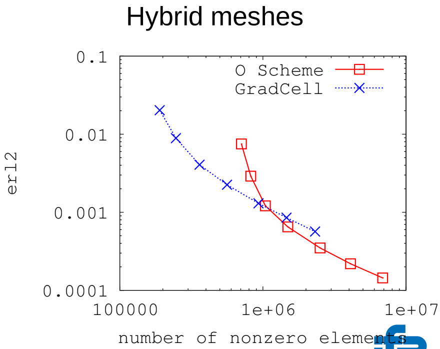
Near well hybrid meshes





Convergence for near well hybrid meshes with $\nu = \mathrm{diag}\left(1, 1, 0.2\right)$ and deviated well analytical solution





Conclusion

- The O scheme lacks robustness for meshes with high aspect ratio (or anisotropy) combined with distorsion
- The Sushi scheme is more robust than the O scheme thanks to its unconditional coercivity
- The GradCell scheme is robust but less accurate, it is much sparser on tetrahedral meshes than the O or Sushi schemes

References

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