Geometric relativity

Joint work with T. Blossier and F.O. Wagner

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Introduction

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This follows from:

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- Use of the group configuration.

Observe that for $A \subset B$ alg. closed (in DCF_0) and any tuple c

$$c \underset{A}{\overset{DCF_0}{\downarrow}} B \iff \mathrm{acl}^{DCF_0}(Ac) \underset{A}{\overset{ACF_0}{\downarrow}} B$$

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Definition

T (with E.I) is 1-based if for $A \subset B$ alg. closed and any c

$$c \mathop{\smile\!\!\!\!\!\!\mid}_{A}^{T} B \iff \mathrm{acl}^{T}(Ac) \mathop{\smile\!\!\!\!\!\!\mid}_{A}^{=} B$$

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- For A algebraically closed and $b \mathop{\smile\!\!\!\!\!\mid}_A c$ then
$$\langle Abc \rangle \subseteq \operatorname{acl}_0(\langle Ab \rangle, \langle Ac \rangle).$$
- If $\bar{a} \in \operatorname{acl}_0(A)$ then $\langle \operatorname{acl}(\bar{a}), A \rangle \subseteq \operatorname{acl}_0(\operatorname{acl}(\bar{a}), \langle A \rangle)$.

Relative 1-based

Definition

T is relatively 1-based over T_0 w.r.t. $\langle\rangle$ if for any $A \subset B$ algebraically closed and any c, if

$$\langle A\bar{c}\rangle \mathop{\downarrow}^{0}_{A} B \text{ , then } c \mathop{\downarrow}^{T}_{A} B.$$

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Remark

Relative 1-basedness is preserved under adding or removing parameters.

Main Lemma

Lemma

Let G be a connected T-definable group, a and b generics with $c = a \cdot b$. Consider a countable Morley sequence D for the generic type over a, b and define

$$\alpha = \mathrm{acl}_0(\mathrm{acl}(b, D), \mathrm{acl}(c, D)) \cap \mathrm{acl}(a, D)$$
$$\beta = \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(c, D)) \cap \mathrm{acl}(b, D)$$
$$\gamma = \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(b, D)) \cap \mathrm{acl}(c, D).$$

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$$\begin{aligned} \alpha &= \operatorname{acl}_0(\operatorname{acl}(b, D), \operatorname{acl}(c, D)) \cap \operatorname{acl}(a, D) \\ \beta &= \operatorname{acl}_0(\operatorname{acl}(a, D), \operatorname{acl}(c, D)) \cap \operatorname{acl}(b, D) \\ \gamma &= \operatorname{acl}_0(\operatorname{acl}(a, D), \operatorname{acl}(b, D)) \cap \operatorname{acl}(c, D). \end{aligned}$$

Then α, β and γ are pairwise independant and each one is 0-algebraic over the other two. Moreover

$$\operatorname{acl}(b, D), \operatorname{acl}(c, D) \underset{\alpha}{\overset{0}{\downarrow}} \operatorname{acl}(a, D).$$

Homogenies

Proposition

Any type-definable group G in T admits a type-definable homogeny S (with parameters) to a T_0-interpretable group H such that given g, g' in G generic independent and h in H such that $S(gg', \bar{h})$, we have

$$\mathrm{acl}(g), \mathrm{acl}(g') \mathop{\smile\!\!\!\!\!\!\;|}^{0}_{h} \mathrm{acl}(gg')$$

and h is 0-interalgebraic with $\mathrm{acl}_0(\mathrm{acl}(g), \mathrm{acl}(g')) \cap \mathrm{acl}(gg')$.

Homogenies II

Theorem

If T is relatively 1-based over T_0 with respect to $\langle\rangle$, then every type-definable group G is monogenous into some T_0-interpretable group H.

Why this notion?

In order to show that the new s.m. set disproved the trichotomy conjecture, Hrushovski introduced the notion of CM-triviality, which prohibits the existence of infinite definable fields (and even bad groups!)

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Definition

T is CM-trivial over T_0 w.r.t. $\langle\rangle$ if for all alg. closed sets $A \subset B$ and every tuple c, if

$$\langle Ac \rangle \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} B,$$

then $\mathrm{Cb}(\bar{c}/A)$ is algebraic over $\mathrm{Cb}(\bar{c}/B)$.

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- If T has E.I. and is CM-trivial over equality w.r.t. acl, then T is CM-trivial.
- If T is rel. 1-based over T_0 w.r.t. $\langle\rangle$, it is CM-trivial over T_0 w.r.t. $\langle\rangle$.
- A Fraïssé-Hrushovski amalgam is CM-trivial over the base data w.r.t. the self sufficient closure.

Some Results

Theorem

Let T be CM-trivial over T_0 w.r.t. $\langle\rangle$. Every connected type-definable group G in T allows a homogeny S to some T_0-interpretable group H such that the $\ker(S)$ is contained (up to finite index) in $Z(G)$ (i.e. $\ker(S)^0 \subseteq Z(G)$)

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Corollary

Any type-definable field in T is definably isomorphic to a subfield of a T_0-interpretable one.

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Corollary

Any type-definable field in T is definably isomorphic to a subfield of a T_0-interpretable one.If T has finite rank, then it is definably isomorphic to a T_0-interpretable field.

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What is then next?

Flatness

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Stay tuned!!