MULTIDIMENSIONAL NUMERICAL RANGE

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1. Notation

Let

- $\mathbb{R}^\infty$ be the linear space of real sequences;
- $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^m$, where $m = 1, 2, \dots, \infty$;
- if $\Omega \subset \mathbb{R}^m$ then $\operatorname{ex}\Omega$ be the set of extreme points of Ω and $\operatorname{conv}\Omega$ is the convex hull of Ω;
- H be a separable complex Hilbert space, $\dim H = \infty$;
- A be a self-adjoint operator in H and $Q_A[\cdot]$ be its quadratic form with domain $\mathcal{D}(Q_A) := \mathcal{D}(|A|^{1/2})$;
- $\sigma(A)$, $\sigma_{\mathrm{ess}}(A)$, $\sigma_{\mathrm{c}}(A)$ and $\sigma_{\mathrm{p}}(A)$ be its spectrum, essential spectrum, continuous spectrum and point spectrum respectively;
- $\lambda_1, \lambda_2, \dots$ be the eigenvalues of A counted with their multiplicities;
- $\mathrm{N}(\lambda)$ be the multiplicity of the eigenvalue λ; if $\lambda \notin \sigma_{\mathrm{p}}(A)$ then $\mathrm{N}(\lambda) := 0$;
- $\hat\sigma^{\pm}_{\mathrm{ess}}(A)$ be the subsets of $\hat{\mathbb{R}} := [-\infty, +\infty]$ such that
  $$\lambda \in \hat\sigma^{+}_{\mathrm{ess}}(A) \iff \dim \Pi_{(\lambda,\mu)} H = \infty \text{ for all } \mu > \lambda,$$
  $$\lambda \in \hat\sigma^{-}_{\mathrm{ess}}(A) \iff \dim \Pi_{(\mu,\lambda)} H = \infty \text{ for all } \mu < \lambda,$$
  where Π_Λ denotes the spectral projection of the operator A corresponding to the set Λ;
- $\hat\sigma_{\mathrm{ess}}(A) := \hat\sigma^{-}_{\mathrm{ess}}(A) \bigcup \hat\sigma^{+}_{\mathrm{ess}}(A)$.

One can easily see that $\sigma_{\mathrm{ess}}(A) = \mathbb{R} \bigcap \hat\sigma_{\mathrm{ess}}(A)$, $-\infty \notin \sigma^{+}_{\mathrm{ess}}(A)$ and $-\infty \notin \sigma^{-}_{\mathrm{ess}}(A)$. We have $\pm\infty \in \hat\sigma_{\mathrm{ess}}(A)$ if and only if the operator $\pm A$ is not bounded from above.

2. Definitions

Definition 1. If m is a positive integer or $m = \infty$, let

- $\sigma(m, A) \subset \mathbb{R}^m$ be the set of vectors $\mathbf{x} = (x_1, x_2, \dots)$ such that $x_j \in \sigma(A)$ for each j and $\#\{j : x_j = \lambda\} \le \aleph(\lambda)$ for all $\lambda \in \sigma(A) \setminus \sigma_{ess}(A)$;
- $\sigma_p(m, A) \subset \mathbb{R}^m$ be the set of vectors $\mathbf{x} = (x_1, x_2, \dots)$ such that $x_j \in \sigma_p(A)$ for each j and $\#\{j : x_j = \lambda\} \le \aleph(\lambda)$ for all $\lambda \in \sigma_p(A)$;
- $\Sigma(m, A) := \bigcup_{\mathbf{u} \subset \mathcal{D}(Q_A)} Q_A[\mathbf{u}] \subset \mathbb{R}^m$, where the union is taken over all orthonormal subsets $\mathbf{u} := \{u_1, u_2, \dots\} \subset \mathcal{D}(Q_A)$ containing m elements.

We call $\Sigma(m, A)$ the *multidimensional numerical range* of A. If m is finite then each of the sets $\sigma(m, A)$, $\sigma_p(m, A)$ and $\Sigma(m, A)$ is the projection of the corresponding set with $m = \infty$.

Let S be the class of infinite matrices $\mathbf{w}$ with nonnegative entries whose row-sums are equal to 1 and column-sums do not exceed 1. If $\mathbf{x} \in \mathbb{R}^\infty$, let us denote $S_{\mathbf{x}} := \bigcup_{\mathbf{w}} \mathbf{wx}$, where the union is taken over all matrices $\mathbf{w} \in S$ such that $\mathbf{wx}$ is well-defined.

Definition 2. $\mathbf{x} \in \mathbb{R}^\infty$ is said to be a *generating sequence* of the operator A if $\Sigma(\infty, A) = S_{\mathbf{x}}$.

Definition 3. If $m < \infty$, let $\mathfrak{T}_A^{(m)}$ be the standard Euclidean topology on $\mathbb{R}^m$. If $m = \infty$, let $\mathfrak{T}_A^{(m)}$ be

- the topology of element-wise convergence on $\mathbb{R}^\infty$ whenever A is unbounded;
- the Mackey topology on l^∞ wherever A is bounded but not compact;
- the Mackey topology on the Marcinkiewicz space generated by the weight sequence $\{|\lambda_1|, |\lambda_2|, \dots\}$ whenever A is compact but not from the trace class;
- the l^1-topology if A belongs to the trace class.

One can prove that in each case $\Sigma(m, A)$ is a subset of the corresponding linear space. Further on the bar denotes the (sequential) $\mathfrak{T}_A^{(m)}$-closure.

3. Main results

Theorem 1. Let $\mathbf{x} \in \mathbb{R}^\infty$. Assume that

(a) either $\sigma_c(A) = \emptyset$, $\mathbf{x} \subset \sigma_p(\infty, A)$ and $\mathbf{x}$ contains all the eigenvalues λ_j of A according to their multiplicities;

(b) or $\sigma_c(A) \neq \emptyset$ and $\mathbf{x}$ coincides with the union of three disjoint subsequences, one of which is defined as above and the other two lie in the open interval $(\inf \sigma_c(A), \sup \sigma_c(A))$ and converge to $\inf \sigma_c(A)$ and $\sup \sigma_c(A)$ respectively.

Then $\mathbf{x}$ is a generating sequence of the operator A.

The following results hold for each $m = 1, 2, \ldots, \infty$.

Theorem 2. The set $\Sigma(m, A)$ is convex, $\operatorname{ex} \Sigma(m, A) \subset \sigma_p(m, A)$ and $\operatorname{ex} \overline{\Sigma(m, A)} \subset \sigma(m, A) \subset \overline{\Sigma(m, A)} = \overline{\operatorname{conv}}\, \sigma(m, A)$.

Theorem 3. $\mathbf{x} \in \operatorname{ex} \Sigma(m, A)$ if and only if there is an interval $[\mu^-, \mu^+] \subset \hat{\mathbb{R}}$ such that

(1) $\mathbf{x}$ consists of all the eigenvalues $\lambda_j \notin [\mu^-, \mu^+]$;

(2) $\sigma_c(A) \subset [\mu^-, \mu^+]$;

(3) $\hat{\sigma}^-_{\mathrm{ess}}(A) \bigcap [-\infty, \mu^-) = \emptyset$ and $\hat{\sigma}^+_{\mathrm{ess}}(A) \bigcap (\mu^+, +\infty] = \emptyset$.

Theorem 4. $\mathbf{x} \in \operatorname{ex} \overline{\Sigma(m, A)}$ if and only if there is an interval $[\mu^-, \mu^+] \subset \hat{\mathbb{R}}$ such that

(1′) $\mathbf{x}$ consists of all the eigenvalues $\lambda_j \notin [\mu^-, \mu^+]$;

(2′) $\hat{\sigma}_{\mathrm{ess}}(A) \subset [\mu^-, \mu^+]$.

Corolary 1. Denote by $\Lambda_e(A)$ the intersection of all intervals $[\mu^-, \mu^+]$ satisfying (2) and (3). We have $\operatorname{ex} \Sigma(m, A) = \emptyset$ whenever $\#\{j : \lambda_j \notin \Lambda_e(A)\} < m$.

Corollary 2. If $n < \infty$ then $\overline{\Sigma(n, A)}$ is a convex polytope and $\Sigma(n, A)$ is a convex subset of the polytope $\overline{\Sigma(n, A)}$ such that $\operatorname{ex} \Sigma(n, A) \subset \operatorname{ex} \overline{\Sigma(n, A)}$.

Example 1. Let $\sigma_c(A) = \emptyset$ and $\mathbf{x}$ be the sequence formed by all the eigenvalues of A. Assume that $\mathbf{x}$ has two accumulation points $\lambda^\pm$. If at least one of these points is not an accumulation point of the sequence $\mathbf{x} \bigcap [\lambda^-, \lambda^+]$ then $\Lambda_e(A) = \emptyset$ and $\mathbf{x}$ is an extreme point of $\Sigma(\infty, A)$.

Example 2. Let A be a semibounded operator, $\sigma_{ess}(A) = [\lambda, -\infty)$ and μ_j be the eigenvalues of A lying in the interval $(-\infty, \lambda)$. Then $\mathbf{x} \in \sigma_p(\infty, A)$ belongs to $\mathrm{ex}\,\Sigma(\infty, A)$ if and only if $\mathbf{x}$ consists of all the eigenvalues μ_j and a collection of entries λ whose number does not exceed $\aleph(\lambda)$. A sequence $\mathbf{x} \in \sigma(\infty, A)$ belongs to $\mathrm{ex}\,\overline{\Sigma(\infty, A)}$ if and only if it consists of all the eigenvalues μ_j and an arbitrary collection of entries λ.

4. Variational formulae

Let $\Omega \subset \mathbb{R}^m$ is a convex set and $\psi : \Omega \to \hat{\mathbb{R}}$ be a function on Ω. Recall that ψ is said to be *quasiconcave* if

$$\psi(\alpha\,\mathbf{x} + (1-\alpha)\,\mathbf{y}) \;\geq\; \min\{\psi(\mathbf{x}), \psi(\mathbf{y})\}, \qquad \forall \alpha \in (0,1),$$

and *strictly quasiconcave* if the left hand side of is strictly greater than the right hand side. The function ψ is quasiconcave if and only if the sets $\{\mathbf{x} \in X : \psi(\mathbf{x}) > \lambda\}$ are convex for all $\lambda \in \hat{\mathbb{R}}$. The function ψ is said to be (sequentially) upper *semicontinuous* if the sets $\{\mathbf{x} \in X : \psi(\mathbf{x}) \geq \lambda\}$ are (sequentially) closed.

The following two corollaries are immediate consequences of Theorem 2.

Corollary 3. We have $\inf\limits_{\mathbf{x}\in\sigma(m,A)} \psi(\mathbf{x}) = \inf\limits_{\mathbf{x}\in\Sigma(m,A)} \psi(\mathbf{x})$ for every quasiconcave (sequentially) upper $\mathfrak{T}_A^{(m)}$-semicontinuous function $\psi : \overline{\Sigma(\infty, A)} \to \hat{\mathbb{R}}$.

Corollary 4. Let $\mathbf{x} \in \Sigma(m, A)$ where $m \leq \infty$. Assume that there exists a quasiconcave function $\psi : \Sigma(m, A) \to \hat{\mathbb{R}}$ such that

(a) either $\psi(\mathbf{x}) < \psi(\tilde{\mathbf{x}})$ for all $\tilde{\mathbf{x}} \in \Sigma(m, A)$ distinct from $\mathbf{x}$,

(b) or ψ is strictly quasiconcave and $\psi(\mathbf{x}) \leq \psi(\tilde{\mathbf{x}})$ for all $\tilde{\mathbf{x}} \in \Sigma(m, A)$.

Then $\mathbf{x} \in \sigma_p(m, A)$.

The functions $\psi(\mathbf{x}) = x_1 x_2 \ldots x_n = \exp(\ln x_1 + \ldots \ln x_n)$ and $\psi(\mathbf{x}) = x_1 + x_2 \cdots + x_n$ are strictly quasiconcave and upper semicontinuous on the set $\{\mathbf{x} \in \mathbb{R}^n : x_i > 0, \forall i\}$. Therefore the above corollaries imply the usual variational formulae for the sum and product of the first n eigenvalues.

5. Remarks and references

There has been an extensive study of various problems related to the numerical range of operators which belong to a given operator algebra. An overview of results obtained in this direction can be found in [1]. Most of them refer to various properties of the corresponding operator algebra. Our approach is very different as it deals not with an operator algebra but with one individual operator A. The idea is to 'pull out' the usual numerical range into higher dimension and investigate the relation between this new multidimensional object and other properties of the operator A. The consideration of the traditional one-dimensional numerical range $\Sigma(1,A)$ is not always sufficient; for instance, it does not give any information about the structure of the spectrum inside $\operatorname{conv}\sigma(A)$. Its multidimensional analogue $\Sigma(m,A)$ is an equally simple object from the theoretical and (if m is not too large) numerical standpoints, which controls more subtle properties A.

There are other concepts of multidimensional numerical range such as the matrix m-numerical range [1] or the quadratic numerical range associated with a given block representation of A [4]. The former is a much more complicated set than $\Sigma(m,A)$, and the latter is not unitary invariant. In our opinion, $\Sigma(m,A)$ is the most natural and straightforward multidimensional generalization of $\Sigma(1,A)$.

If $A \neq A^*$ and $m \geq 2$ then the set $\Sigma(m,A)$ may not be convex (even if the operator A is normal) and a little is known about its geometric structure. In [2] the authors proved that $\overline{\operatorname{conv}}\,\sigma_p(m,A) = \overline{\operatorname{conv}}\,\Sigma(m,A)$ whenever A is a normal $m \times m$-matrix. There are also some results on the so-called c-numerical range of a finite matrix A, which is defined as the image of $\Sigma(m,A)$ under the map $\mathbf{x} \mapsto \langle \mathbf{x}, c\rangle \in \mathbb{C}$ where c is a fixed m-dimensional complex vector (see [3], [5], [6]).

Proofs of Theorems 1–4 and other relevant results can be found in [7]. Offprints are available on request.

References

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