SPECTRAL SINGULARITIES AND ASYMPTOTICS OF CONTRACTIVE SEMIGROUPS

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Theorem (Sz.-Nagy - Foias - Gohberg - Kreĭn). *Let L be a maximal dissipative operator such that $\sigma(L) \subset \mathbb{R}$. Then the following are equivalent*

(i) L is similar to a self-adjoint operator[1]*;*

(ii) The resolvent of L satisfies the estimate

$$\left\|(L-z)^{-1}\right\| \leq \frac{C}{\operatorname{Im} z}, \qquad \operatorname{Im} z > 0; \tag{1}$$

(iii) L generates a uniformly bounded group $Z_t = e^{iLt}$: $\sup_{t<0} \|Z_t\| < \infty$.

For a dissipative operator, L, with absolutely continuous spectrum (see Glossary), $E \in \mathbb{R}$ is *a spectral singularity* [2] if

$$\limsup_{z \searrow E} \left(\operatorname{Im} z \left\|(L-(E+i\varepsilon))^{-1}\right\|\right) = \infty.$$

Thus, a bounded dissipative operator L with absolutely continuous spectrum has no spectral singularities if and only if the semigroup $Z_t = e^{iLt}$, $t < 0$, is bounded.

General problem: analyze in detail the impact of spectral singularities on asymptotic behaviour of Z_t as $t \to -\infty$.

Specific task: the problem of localization of spectral singularities in terms of the asymptotic in the "simplest" case of finitely many singularities of finite power order, that is, of calculating their orders and locations from the asymptotics of Z_t.

[1] that is, there exists a bounded operator U with bounded inverse such that $L = UAU^{-1}$ for some self-adjoint A.

Background: In the self-adjoint theory, if a self-adjoint operator, A, has an absolutely continuous spectrum and finitely many eigenvalues, $\{\lambda_j\}$, then the similar problem of calculating the λ_j's from the asymptotic of e^{iAt} is solved trivially because of the spectral theorem. Namely, we fix arbitrary u and v such that $(u, e_j) \neq 0$, $(v, e_j) \neq 0$ for all e_j, the eigenvectors of A, and consider $f(t) = (e^{iAt}u, v)$. Then

$$f(t) = \int e^{i\lambda t} d\rho(t) + \sum_{j=0}^{N} c_j e^{i\lambda_j t},$$

where ρ is an absolutely continuous measure, and $c_j = (u, e_j)(e_j, v) \neq 0$ for all j. The first term is $o(1)$ as $t \to \pm\infty$ by the Riemann-Lebesgue lemma, and one can determine the λ_j's from the finite sum in the right hand side, provided that we know $f(t)$ at large t.

The nonself-adjoint problem above is considerably more complicated since for an operator with spectral singularities we do not have a spectral decomposition *converging in the topology of the original Hilbert space* [8]. This makes it natural to consider first the case of finitely many spectral singularities. The analysis of an operator with isolated power singularities still constitutes a highly nontrivial problem, which has important applications. For instance, the Schrödinger operator on the real axis with complex potential decreasing like $O\left(\exp(-C\sqrt{x})\right)$ studied in [4], is of the class under consideration.

Previously known:

In the following theorem L is a maximal dissipative operator with real spectrum satisfying $\left\|(L-z)^{-1}\right\| \leq (\operatorname{Im} z - \omega)^{-1}$ for some $\omega > 0$ and all z with $\operatorname{Im} z > \omega$. The latter condition guarantees that $Z_t = e^{iLt}$ exsists for all t as a bounded operator.

Theorem 1. [1] *Assume that the estimate*

$$\left\|(L-z)^{-1}\right\| \leq C\left(\operatorname{Im} z\right)^{-p-1} \tag{2}$$

holds for all z in a strip $0 < \operatorname{Im} z < \varepsilon_0$. *Then*

$$\left\|Z_t^{-1}\right\| \leq C'\left(1 + t^p\right). \tag{3}$$

for all $t \geq 0$.

Remark: In fact this theorem can be generalized so as to allow any monotone function instead of power, see [5].

I. RESULTS (informal description).

- Sharpness of the upper estimate (3) in the power scale.
- An upper estimate for local norm of $P_\omega Z_{-t}$ where P_ω is the spectral projection of the real part of the operator L corresponding to the interval $\omega \subset \mathbb{R}$.
- An algorithmic solution of the localization problem for singularities of the orders greater than $p - 1/2$ where p is the maximal order, in terms of the asymptotic of Z_t as $t \to -\infty$.

- An application: the three - dimensional Boltzmann operator [6].

II. RESULTS (precise formulations).

The definition of what we call a spectral singularity of the real order p is best given in terms of the boundary behaviour of the characteristic function [3] of the operator.

We assume throughout that L is a maximal dissipative operator in a Hilbert space H with bounded imaginary part $V = \operatorname{Im} L$ such that $\sigma_{ess}(L) \subset \mathbb{R}$. Denote by E the subspace $E = \overline{\operatorname{Ran} V} \subset H$. By the *characteristic function* of L we call the contractive analytic function $S(z)\colon E \to E$, $z \in \mathbb{C}_+$, defined by the formula

$$S(z) = I + 2i\sqrt{V}\,(L^* - z)^{-1}\,\sqrt{V}.$$

The set $\mathbb{C}_+ \bigcap \rho(L)$ coincides with the set of such $z \in \mathbb{C}_+$ that the operator $S(z)$ has bounded inverse defined on the whole of E. Moreover, the following estimate is valid,

$$\|S^{-1}(z)\| \asymp \operatorname{Im} z \left\|(L - z)^{-1}\right\|. \tag{4}$$

This estimate shows that for an operator with absolutely continuous spectrum $E \in \mathbb{R}$ is a spectral singularity if and only if $\limsup_{z \searrow E} \|S^{-1}(z)\| = \infty$.

In applications, it often happens that the characteristic function is analytic on the real axis, and $S(k) - I$ is compact, which implies that the spectral singularities are exactly the real poles of S^{-1}. Examples include the Schrödinger operator with exponentially decreasing potential [7, 10] and the Friedrichs model with rank 1 analytic perturbation, and the one-velocity transport operator considered below. The following definition provides the simplest natural abstract generalization of this situation.

A point $k_0 \in \mathbb{R}$ is said to be a *spectral singularity of the order* $p > 0$, $p \in \mathbb{R}$, *in the strict sense*, if for some nonzero $e_0 \in E$

$$\|S(k)e_0\|_E \le C\,|k - k_0|^p\,, \tag{5}$$

$$\|S^{-1}(k)\| \le C\,|k - k_0|^{-p} \tag{6}$$

for a. e. k in a vicinity of k_0 on the real axis.

The "in the strict sense" clause is inserted in this definition to indicate that it is not merely the requirement of exactness of the estimate (6): the vector e_0 in (5) does not depend on k.

We now proceed to study the local behaviour of the semigroup Z_t with respect to the spectral representation of the real part of the operator L. For an interval $\omega \subset \mathbb{R}$ define P_ω to be the spectral projection of $A = \operatorname{Re} L$ for the ω.

Introduce the following

Assumption P. The dissipative operator L has at most finitely many spectral singularities, k_j, such that for each k_j there exists a $p_j > 0$ such that the estimate

$$\left\|S^{-1}(k)\right\| \le C\,|k - k_j|^{-p_j}$$

holds for a. e. k in a vicinity of k_j on the real axis, and $\operatorname{ess\,sup}_{|k|>M}\|S^{-1}(k)\|$ is finite for some M (no spectral singularity at infinity).

Let $p = \max_j p_j$.

Corollary 2. *If the assumption P is satisfied, then for all $t > 0$*

$$\left\| Z_{-t}|_{\mathcal{N}_e} \right\| \leq C\,(1 + t^p)$$

with some $C > 0$.

The proof consists in verification of the assumption of theorem 1 on the basis of a uniqueness theorem.

Theorem 3. *Let the assumption P be satisfied with $p \neq 1/2$, and let $\sqrt{V}$ is* $\operatorname{Re} L$ *-smooth in the sense of Kato. Then for any closed interval $\omega \subset \mathbb{R}$ we have*

$$\left\| P_\omega Z_{-t}|_{\mathcal{N}_e} \right\| \leq C_\omega \left(1 + t^{p-1/2} + \sum_{j:k_j \in \omega} t^{p_j} \right), \quad t > 0. \tag{7}$$

In particular,

$$\left\| P_\omega Z_{-t}|_{\mathcal{N}_e} \right\| \leq C_\omega \left(1 + t^{p-1/2} \right), \quad t > 0, \tag{8}$$

if ω does not contain spectral singularities.

Remark. (8) holds for $p \geq 1$ and an interval ω not containing the singularities without the smoothness assumption.

Hints on the proof. The proof is independent of the functional model and combines three ingredients:

1. An integral representation of $Z_{-t}u$ via smoothed resolvent through Doimelle formula.

2. The following fundamental identity, first established in [9]. Let D be an arbitrary dissipative operator with a bounded imaginary part V, and $\alpha = \sqrt{2\,V}$. Then for all $\varepsilon > 0$ and $u \in H$

$$\int_{\mathbb{R}} \left\| \alpha\,(D - k + i\varepsilon)^{-1}\,u \right\|^2 dk \leq \pi\,\|u\|^2. \tag{9}$$

3. Transformation of integration contours and an elementary case of the Carleson embedding theorem.

The problem of detecting the spectral singularities can now be formulated as follows. Suppose we are given an operator with finitely many spectral singularities, each of a finite power order in the strong sense. How to calculate their orders, p_j, and locations, k_j, in terms of the norms $\|P_\omega Z_{-t}\|$? To advance in this direction we need results on exactness of the upper estimate (7).

Theorem 4. *Assume L has a spectral singularity of order p in the strict sense. Then for any sufficiently small $\varepsilon > 0$ there exists such a $u \in \mathcal{N}_e$ that*

$$\|Z_{-t}u\| = t^{p-\varepsilon}(1 + o(1)), \quad t \to +\infty. \tag{10}$$

The proof consists in an explicit construction of the required Cauchy data u for the Szökefalvi- Nagy - Foias functional model of the operator. In the following theorem we give an asymptotic expression for the evolution of the vector u.

Theorem 5. *Assume L is a dissipative operator, and $\sqrt{V}$ is $\operatorname{Re} L$ - smooth. Then for any spectral singularity $k_0 \in \mathbb{R}$ of a real order $n > 1/2$ and any $\varepsilon > 0$ small enough there exists a $u \in \mathcal{N}_e$ satisfying (10) and such that for any closed interval ω, $k_0 \notin \omega$,*

$$P_\omega Z_{-t} u = e^{-ik_0 t} t^{n-1/2-\varepsilon} \Psi_{k_0} + O\left(t^{\max\{0, n-3/2-\varepsilon\}}\right) \tag{11}$$

where $\Psi_{k_0} = (A_\omega - k_0)^{-1} P_\omega \alpha e_0$, $A_\omega = AP_\omega$, e_0 is defined in (5), and the O-symbol refers to the norm in H.

Example (Friedrichs model). Let I be a compact interval, $H = L^2(I)$ and a vector $\varphi \in L^\infty(I)$ is given. Define $L = A + i\langle \cdot, \varphi\rangle\varphi$ where $(Af)(s) = sf(s)$. Then the smoothness assumption of theorem 5 is satisfied, and the asymptotics (11) takes in this case the form

$$Z_{-t} u = e^{-ik_0 t} t^{n-1/2-\varepsilon} \Psi_{k_0} + r_t$$

where

$$\Psi_{k_0}(s) = \frac{\varphi(s)}{s - k_0},$$

the function r_t satisfies

$$\left(\int_{|s-k_0|>\delta} |r_t(s)|^2 ds \right)^{1/2} \leq C_\delta \left(1 + t^{n-3/2-\varepsilon}\right)$$

for any $\delta > 0$, and the equality holds for each $t > 0$ for a. e. $s \in I$. The function Ψ_{k_0} makes sense of a regular part of the generalized eigenfunction of L corresponding to the point k_0: any formal solution to $Lu = k_0 u$ coincides with Ψ_{k_0} for $s \neq k_0$. Thus, the improper eigenfunction of the spectral singularity appears as a leading term coefficient in the asymptotic expansion outside arbitrary small vicinity of the singularity in the spectral representation of the real part of the operator. This matches the expectation coming from consideration of the discrete spectrum case, and can be considered as a rigorous justification of the idea that spectral singularity is a kind of resonance which has been drawn into the continuous spectrum.

This theorem shows, in particular, that the contribution of arbitrary small neighborhood of the spectral singularity in the asymptotic of $Z_{-t}u$ is of the maximal order, $t^{n-\varepsilon}$, in norm.

Corollary. *If $p = \max_j p_j \geq 1/2$, then the estimate (8) is exact in the power scale, that is,*

$$\limsup_{t\to+\infty} \frac{\ln \left\| P_\omega Z_{-t}|_{\mathcal{N}_e} \right\|}{\ln t} = p - \frac{1}{2}$$

for any interval ω and any operator L of the class under consideration.

Combining the results obtained, we now give a solution to the localization problem for singularities of higher orders. First, we determine the maximal order as the least p such that $\left\| Z_{-t}|_{\mathcal{N}_e} \right\| = O\left(t^p\right)$ at large t. Then the following assertion is valid.

Proposition 6. *Let a dissipative operator $L = A + iV$, $V \geq 0$, have at most finitely many spectral singularities, each of a finite power order in the strict sense, and p be the maximal order. Suppose that $\sqrt{V}$ is A - smooth. For a closed interval $\omega \subset \mathbb{R}$ define the number $p(\omega) = \inf_{s \geq 0}\{s : \left\| P_\omega Z_{-t}|_{\mathcal{N}_e} \right\| \leq C_s\left(1 + t^s\right)\}$. Then*

(i) ω contains a singularity of the order $p(\omega)$ if $p(\omega) > p - 1/2$, provided that $p(\omega) \neq 0$,

(ii) The interior of ω does not contain a singularity of an order greater than $p(\omega)$.

This result allows to localize singularities of orders greater than $\max\{0, p - 1/2\}$. The separation of singularities of lower orders cannot be achieved in this way since, as it is seen from (11), the norm estimate for an interval containing such singularities is determined by the contribution of singularities outside the interval.

III. EXAMPLE: transport operator. Let $d\Omega_p$ be the Lebesgue measure on the unit sphere $\mathbb{S}^2$. Given a nonnegative compactly supported function $c \in L^\infty\left(\mathbb{R}^3\right)$, define the operator, T, in the Hilbert space $H = L^2(\mathbb{R}^3 \times \mathbb{S}^2)$ by the formula ($q \in \mathbb{R}^3, \widehat{p} \in \mathbb{S}^2$)

$$T = i\widehat{p} \cdot \nabla_q + ic(q)K, \quad K = \frac{1}{4\pi}\int_{\mathbb{S}^2} \cdot \, d\Omega_p$$

on the natural domain of its real part. This operator arises in the one - speed neutron transport theory, see [6, 11] for details. Then T is a maximal dissipative operator, and the essential spectrum of T coincides with the real axis. Let H_{ess} be the invariant subspace of T corresponding to $\sigma_{ess}(T)$. Define the set $\mathcal{E} \subset L^\infty\left(\mathbb{R}^3\right)$ as $\mathcal{E} = \{c \ : \ \ker(I + Q(0)) \neq 0\}$ where $Q(0)$ is the integral operator in $L^2(\mathbb{R}^3)$ with the kernel $-\frac{1}{4\pi}\sqrt{c(q)}\frac{1}{|q'-q|^2}\sqrt{c(q')}$.

Theorem. *[6] The completely nonself-adjoint part of the restriction, T_{ess}, of T to H_{ess} has absolutely continuous spectrum and satisfies the condition of corollary 2. If $c \notin \mathcal{E}$ then the operator T_{ess} is similar to a self-adjoint operator. If $c \in \mathcal{E}$ then the operator T has the unique spectral singularity of the order* 1 *located at the point* 0.

Corollary. [6] *Let $Z^e_t = \exp(-iTt)|_{H_{ess}}$, $t \geq 0$. If $c \in \mathcal{E}$ then*

$$\|Z^e_t\| \leq C(1 + t),$$

and this estimate is exact in the sense given by theorem 3.

GLOSSARY

The notation $z \searrow E$ for real E means that z tends to E from the upper half plane.

Let H' be the minimal reducing subspace of L containing E, then $L_{pure} := L|_{H'}$ is *the completely nonselfadjoint part* of the operator L. Let $H^2_\pm(E)$ stand for the Hardy classes of E-valued functions f analytic in $\mathbb{C}_\pm$, respectively, and such that $\sup_{\varepsilon>0}\int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2_E \, dk$ is finite.

The absolutely continuous subspace $\mathcal{N}_e$ of the operator L_{pure} [2, 9] is defined as follows,

$$\mathcal{N}_e = \overline{\widetilde{\mathcal{N}_e}}, \qquad \widetilde{\mathcal{N}_e} \equiv \left\{u \in H' : \sqrt{V}\left(L - z\right)^{-1} u \in H^2_+(E)\right\}.$$

Elements of the linear set $\widetilde{\mathcal{N}_e}$ are called *smooth vectors* of the operator L.

If L is completely nonself-adjoint ($L = L_{pure}$), then it is referred to as having *absolutely continuous spectrum* if $H = \mathcal{N}_e$.

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