Distribution of lattice points in Euclidean and hyperbolic spaces

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Let Γ be a lattice in $\mathbb{R}^d$, $F = \mathbb{R}^d/\Gamma$, $\text{vol}(F) = 1$. For $q \in F$ and $\rho > 0$ denote $B(q,\rho)$ the ball centered at q of radius ρ and

$$N(q,\rho) = \#\{\gamma \in (\Gamma \cap B(q,\rho))\},$$

Describe the behaviour of $N(q,\rho)$ for large ρ.

Obvious:

$$N(q,\rho) \sim \text{vol}(B(q,\rho)) = \omega_d \rho^d.$$

Denote $R(q,\rho) := N(q,\rho) - \omega_d \rho^d$.

Gauss:

$$R(q,\rho) = O(\rho^{d-1}).$$

$d = 2$. Sierpiński:

$$R(q,\rho) = O(\rho^{2/3}).$$

Conjecture: $\Gamma = \mathbb{Z}^2$. Then

$$R(0,\rho) = O(\rho^{(1/2)-\epsilon}).$$

Hardy, Littlewood, Walfisz, Cheng, Huxley:

$$R(0,\rho) = O(\rho^{(46/73)}).$$

Arbitrary d. Landau:

$$R(q,\rho) = O(\rho^{d-2+2/(d+1)}).$$

Walfisz, Bentkus, Götze. Let $d \geq 5$. Then

$$R(q,\rho) = O(\rho^{d-2})$$

(precise estimate).

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Average value of R. Denote

$$\sigma_j(\rho) := \left(\int_P |R(q,\rho)|^j dq\right)^{1/j}, \qquad j = 1,2$$

Theorem(A.Sobolev, L.Parnovski)

$$\rho^{\frac{d-1}{2}-\epsilon} \ll \sigma_j(\rho) \ll \rho^{\frac{d-1}{2}};$$

$$\rho^{\frac{d-1}{2}} \ll \sigma_j(\rho) \qquad \text{iff } d \not\equiv 1 (mod 4)$$

$(j = 1,2)$

Similar question for the annulus. Let $B(q,\rho,\eta) = \{x \in \mathbb{R}^d : \rho < |x-q| < \rho+\eta\}$, $\rho \to \infty$, $\eta = \eta(\rho)$ is a continuous function. As before,

$$N(q,\rho,\eta) = \#\{\gamma \in (\Gamma \cap B(q,\rho,\eta))\},$$

$$R(q,\rho,\eta) = N(q,\rho,\eta) - |B(q,\rho,\eta)|,$$

$$\sigma_j(\rho,\eta) := \left(\int_F |R(q,\rho,\eta)|^j dq\right)^{1/j}, \qquad j = 1,2.$$

Theorem(R.Hill, L.Parnovski)

(i) Let $\eta \asymp 1$. Then

$$\rho^{\frac{d-1}{2}-\varepsilon} \ll \sigma_j(\rho,\eta) \ll \rho^{\frac{d-1}{2}};$$

$$\rho^{\frac{d-1}{2}} \ll \sigma_j(\rho,\eta) \qquad \text{iff } d \not\equiv 3 (mod 4).$$

(ii) Let $\eta \to \infty$, $\frac{\eta}{\rho} \to 0$. Then

$$\rho^{\frac{d-1}{2}}\eta^{-\varepsilon} \ll \sigma_j(\rho,\eta) \ll \rho^{\frac{d-1}{2}};$$

$$\rho^{\frac{d-1}{2}} \ll \sigma_j(\rho,\eta) \qquad \text{never.}$$

(iii) Let $\eta \asymp \rho$. Then

$$\rho^{\frac{d-1}{2}-\varepsilon} \ll \sigma_j(\rho,\eta) \ll \rho^{\frac{d-1}{2}};$$

$$\rho^{\frac{d-1}{2}} \ll \sigma_j(\rho,\eta) \qquad \text{iff } d \not\equiv 1 (mod 4).$$

(iv) Let $\eta \to 0$. Then

$$\rho^{\frac{d-1}{2}-\varepsilon}\eta^{\frac{1}{2}+\varepsilon} \ll \sigma_2(\rho,\eta) \ll \rho^{\frac{d-1}{2}}\eta^{\frac{1}{2}}.$$

If $\eta \asymp (\ln\ln\rho)^{-1}$, then

$$\rho^{\frac{d-1}{2}}\eta^{\frac{1}{2}} \ll \sigma_2(\rho,\eta) \qquad \text{iff } d \not\equiv 3 (mod 4).$$

If $\eta \asymp \rho^{\alpha}$, $\alpha > 0$, then

$$\rho^{\frac{d-1}{2}}\eta^{\frac{1}{2}} \ll \sigma_2(\rho,\eta) \qquad \text{always.}$$

Hyperbolic lattices

Let Γ be a discrete group of isometries of the hyperbolic space $\mathbb{H}^d$, $F = \mathbb{H}^d/\Gamma$. We assume $\mathrm{vol}(F) < \infty$, but F can be non-compact. For any $p \in F$ corresponding lattice is $\Gamma p = \{\gamma p : \gamma \in \Gamma\}$.

$$N(\rho, q, p) \doteq \#\{\Gamma p \cap B(q, \rho)\}.$$

Is $N(\rho, q, p) \sim \frac{\mathrm{vol}(B(q,\rho))}{\mathrm{vol}(F)}$?

Yes (Delsarte)

Put $R(\rho, q, p) = N(\rho, q, p) - \frac{\mathrm{vol}(B(q,\rho))}{\mathrm{vol}(F)}$. Estimates of R and further terms(!): Huber, Patterson, Selberg, Lax-Phillips, Levitan.

Notation: Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be eigenvalues of $-\Delta$ acting on F, ϕ_j be corresponding orthonormal eigenfunctions. Let N be such that $\lambda_N < \frac{(d-1)^2}{4} \leq \lambda_{N+1}$. Denote $\lambda_j = s_j(d-1-s_j)$. If $j \leq N$, then $s_j \in \mathbb{R}$; in this case we choose $s_j > \frac{d-1}{2}$. If $j > N$, then $\Re s_j = \frac{d-1}{2}$. Essential spectrum of $-\Delta$ is either $\emptyset$ (if F is compact) or $[\frac{(d-1)^2}{4}, +\infty)$ (if F is non-compact). In the latter case we assume for simplicity that F has one cusp and denote $E(\cdot, \lambda)$ the eigenfunctions of the continuous spectrum (canonically normalized; they are called *Eisenstein series*).

Theorem(Huber, Patterson, Selberg, Lax, Phillips, Levitan)

$$R(\rho, q, p) = \sum_{j=1}^{N} w(s_j)\phi_j(q)\phi_j(p)e^{s_j\rho} + O\left(\rho^{\frac{3}{d+1}} e^{\left(d-1-\frac{d-1}{d+1}\right)\rho}\right),$$

$$w(s) = \pi^{\frac{d-1}{2}} \frac{\Gamma(s - \frac{d-1}{2})}{\Gamma(s+1)}.$$

The sum is over all j such that $d - 1 - \frac{d-1}{d+1} < s_j < d - 1$. As before, define

$$\sigma_2(\rho, p) := \left(\int_F |R(\rho, q, p)|^2 d\mu(q)\right)^{1/2}.$$

Theorem(R.Hill,L.Parnovski)

$$\sigma_2(\rho, p)^2$$
$$= \sum_{j=1}^{N} \sum_{l \geq 0: s_j - l > (d-1)/2} w(s_j, l) |\phi_j(p)|^2 e^{2(s_j - l)\rho}$$
$$+ \Big(\sum_{j=1}^{N} \sum_{l \geq 0: s_j - l = (d-1)/2} w(s_j, l) |\phi_j(p)|^2$$
$$+ \frac{\pi^{d-1}}{\Gamma(\frac{d+1}{2})^2} \left| E\left(p, \frac{(d-1)^2}{4}\right) \right|^2 \Big) \rho e^{(d-1)\rho}$$
$$+ \sum_{j: s_j = (d-1)/2} |\phi_j(p)|^2 (A\rho^2 + B\rho) e^{(d-1)\rho}$$
$$+ O(e^{(d-1)\rho}).$$