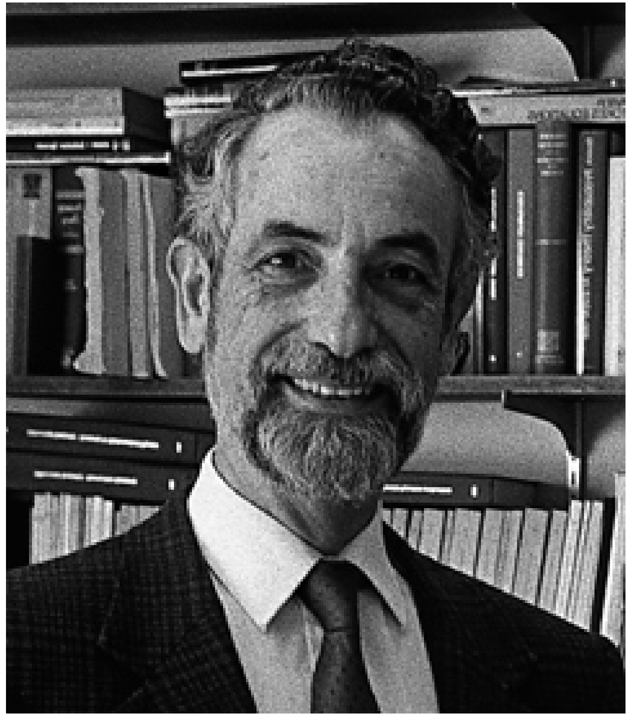
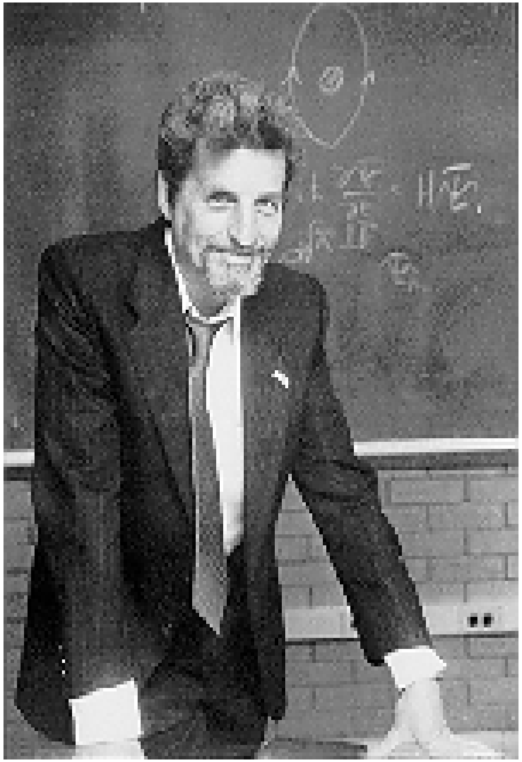
Godfrey Harold Hardy
1877-1947



Elliott Lieb
Princeton University

Karl Menger
1902-1985

“Not that, if one were to spread the insight into the methods of mathematics more widely, this would necessarily result in many more intelligent things being said than today, but certainly many fewer unintelligent things would be said."



Yakir Aharonov

Professor of Tel Aviv University and University of South Carolina

David Bohm
1917-1974

"I would say that in my scientific and philosophical work, my main concern has been with understanding the nature of reality in general and of consciousness in particular as a coherent whole, which is never static or complete but which is an unending process of movement and unfoldment...."

Geometric Hardy Type Inequalities for Many Particles

Ari Laptev

Joint work with Maria and Thomas Hoffmann-Ostenhof and Jesper Tidblom

LMS Durham Symposium,
August 8, 2005

Many thanks to the organizers:

Brian Davies, Yuri Safarov and Eugene Shargorodsky

Notations.

Let $x \in \mathbb{R}^{nN}$, $x = (x_1, x_2, \ldots, x_N)$, $x_j = (x_{j,1}, \ldots, x_{j,n}) \in \mathbb{R}^n$, $r_{ij} = |x_i - x_j|$,

We consider the folowing problem:

Find the best constants $C(n,N)$, such that

$$-\Delta = -\sum_{j=1}^{N} \Delta_{x_j} \geq C(n,N) \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^2},$$

which is a generalization of Hardy's inequality

$$-\Delta \geq \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2}, \quad x \in \mathbb{R}^n.$$

Motivation

Let us consider the following Schrödinger operator in $L^2(\mathbb{R}^{3N})$

$$-\sum_{j=1}^{N} \Delta_{x_j} - \sum_{j=1}^{N} \frac{Z}{|x_j|} + \sum_{1\le k<j\le N} \frac{1}{|x_k - x_j|}.$$

In 1984 E.Lieb obtained an uniform inequality on the number N of electrons that can be bound to an atomic nucleus of charge Z. Namely

$$2Z > N - 1.$$

There is a conjecture $N \sim Z + O(1)$, as $Z \to \infty$.
In the proof Lieb used the standard 3D Hardy inequality

$$-\Delta \ \geq \frac{1}{4} \frac{1}{|x|^2}, \quad x \in \mathbb{R}^3.$$

Lemma. *Let* $u \in C_0^\infty(\mathbb{R}^m)$, $m \geq 1$ *and let*

$$\mathbf{F} = \Big(F_1(x), F_2(x), \dots, F_m(x)\Big)$$

be a vector field in $\mathbb{R}^m$. *Then*

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^m} |u|^2 \operatorname{div} \mathbf{F}\, dx \right)^2}{\int_{\mathbb{R}^m} |u|^2 |\mathbf{F}|^2\, dx}. \tag{1}$$

Proof. We use the Cauchy-Schwarz inequality and partial integration

$$\left| \int_{\mathbb{R}^m} |u|^2 \operatorname{div} \mathbf{F}\, dx \right| = 2 \left| \Re \int_{\mathbb{R}^m} \langle \mathbf{F},\, \nabla u \rangle \bar{u}\, dx \right|$$
$$\leq 2 \Big(\int_{\mathbb{R}^m} |u|^2 |\mathbf{F}|^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^m} |\nabla u|^2 dx \Big)^{1/2}.$$

Squaring this inequality completes the proof.

Remark. Note that one can linearize the inequality (**??**) by using that for positive B, $A^2/B \geq 2A - B$. Namely,

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^m} |u|^2 \operatorname{div} \mathbf{F}\, dx \right)^2}{\int_{\mathbb{R}^m} |u|^2 |\mathbf{F}|^2\, dx}$$

implies

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^m} |u|^2 \Big(2 \operatorname{div} \mathbf{F} - |\mathbf{F}|^2 \Big)\, dx.$$

The same inequality one obtains directly from the identity

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx = \int_{\mathbb{R}^m} \left| \Big(\nabla - \frac{1}{2} \mathbf{F} \Big) u \right|^2 dx + \frac{1}{4} \int_{\mathbb{R}^m} |u|^2 \Big(2 \operatorname{div} \mathbf{F} - |\mathbf{F}|^2 \Big)\, dx.$$

The identity

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx = \int_{\mathbb{R}^m} \left| \left(\nabla - \frac{1}{2}\,\mathbf{F} \right) u \right|^2 dx + \frac{1}{4} \int_{\mathbb{R}^m} |u|^2 \left(2\,\mathrm{div}\,\mathbf{F} - |\mathbf{F}|^2 \right) dx$$

can formally be rewritten as

$$-\Delta = \left(\nabla - \frac{1}{2}\,\mathbf{F} \right)^* \left(\nabla - \frac{1}{2}\,\mathbf{F} \right) + \frac{1}{4}\left(2\,\mathrm{div}\,\mathbf{F} - |\mathbf{F}|^2 \right) \geq \frac{1}{2}\mathrm{div}\,\mathbf{F} - \frac{1}{4}|\mathbf{F}|^2.$$

1D Hardy inequality with N particles.

Theorem. *Let* $u \in H_0^1(\mathbb{R}^N \backslash \mathcal{N}_N)$, *where* $\mathcal{N}_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \,\Big|\, x_i = x_j$ *for some* $i \neq j\}$. *Then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^N} |u|^2 \Big(\sum_{i \neq j} \frac{1}{r_{ij}^2} \Big) dx.$$

Proof. We use

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^N} |u|^2 \Big(2 \operatorname{div} \mathbf{F} - |\mathbf{F}|^2 \Big) dx.$$

choosing

$$\mathbf{F} = -\Big(\sum_{k \neq 1} \frac{1}{x_1 - x_k}, \ldots, \sum_{k \neq N} \frac{1}{x_N - x_k} \Big).$$

Remarks.

- The standard Hardy inequality on $(0, \infty)$ is

$$\int_0^\infty |f'|^2 dx \geq \frac{1}{4} \int_0^\infty \frac{|f|^2}{|x|^2} dx.$$

This implies $-\frac{\partial^2}{\partial^2 x_i} - \frac{\partial^2}{\partial^2 x_j} \geq \frac{1}{4}\left(\frac{1}{r_{ij}^2} + \frac{1}{r_{ji}^2}\right)$. If we add up then we obtain the constant $\frac{1}{4(N-1)}$ rather than $1/4$.

- The constant $1/4$ is optimal.

3D Hardy inequalities with N particles.

Let

$$A_N(x) := \sum_{i \neq j}^{N} \frac{1}{r_{ij}^2}, \qquad B_N(x) := \sum_{j=1}^{N} \sum_{i \neq k,\, i,k \neq j} \frac{(x_j - x_i) \cdot (x_j - x_k)}{r_{ij}^2 r_{jk}^2}.$$

and let $\mathbf{F}$ be the following $3 \times N$ vector

$$\mathbf{F}(x) = \nabla \ln \Pi_{j \neq k} |x_j - x_k| = \Big(\sum_{k \neq 1} \frac{x_1 - x_k}{|x_1 - x_k|^2}, \dots, \sum_{k \neq N} \frac{x_N - x_k}{|x_N - x_k|^2} \Big).$$

Simple calculation shows that

$$\operatorname{div} \mathbf{F}(x) = A_N(x) \quad \text{and} \quad |\mathbf{F}(x)|^2 = A_N(x) + B_N(x).$$

We now define

$$K(N) = \sup_{x \in \mathbb{R}^{3N}} \frac{B_N(x)}{A_N(x)}.$$

Then the main lemma implies

$$\int_{\mathbb{R}^{3N}} |\nabla u(x)|^2\, dx \geq \frac{1}{4}\,\frac{\left(\int_{\mathbb{R}^{3N}} |u(x)|^2 \operatorname{div}\mathbf{F}(x)\, dx\right)^2}{\int_{\mathbb{R}^{3N}} |u(x)|^2\, |\mathbf{F}(x)|^2\, dx}$$

$$= \frac{1}{4}\,\frac{\left(\int_{\mathbb{R}^{3N}} |u(x)|^2 A_N(x)\, dx\right)^2}{\int_{\mathbb{R}^{3N}} |u|^2 A_N(x)(1 + B_N(x) A_N^{-1}(x))\, dx}$$

$$\geq \frac{1}{4 + 4K(N)} \int_{\mathbb{R}^{3N}} |u(x)|^2 A_N(x)\, dx$$

$$= \frac{1}{4 + 4K(N)} \int_{\mathbb{R}^{3N}} |u(x)|^2 \Big(\sum_{i \neq j}^{N} \frac{1}{r_{ij}^2}\Big)\, dx.$$

Thus we obtain the following result:

Theorem. *Let* $u \in H^1(\mathbb{R}^{3N})$*, then*

$$\int_{\mathbb{R}^{3N}} |\nabla u|^2 dx \geq \frac{1}{4 + 4K(N)} \int_{\mathbb{R}^{3N}} |u|^2 \sum_{i \neq j} \frac{1}{r_{ij}^2} \, dx,$$

The problem is now reduced to finding a good estimate for the value of $K(N)$.

Lemma. *Let x_1, x_2, x_3 be three n-dimensional vectors. Let R be the circumradius of the triangle with corners x_1, x_2, x_3. Then*

$$\frac{1}{2R^2} = \frac{(x_1 - x_2) \cdot (x_1 - x_3)}{r_{12}^2 r_{13}^2} + \frac{(x_2 - x_1) \cdot (x_2 - x_3)}{r_{12}^2 r_{23}^2} + \frac{(x_3 - x_1) \cdot (x_3 - x_2)}{r_{13}^2 r_{23}^2}.$$

Proof. Let $a = x_1 - x_2$ and $b = x_1 - x_3$. Then

$$\text{r.h.s.} = \frac{(a \cdot b)}{|a|^2 |b|^2} - \frac{a \cdot (b - a)}{|a|^2 |b - a|^2} - \frac{b \cdot (a - b)}{|b|^2 |b - a|^2}$$
$$= \frac{2\Big(|a|^2 |b|^2 - (a \cdot b)^2\Big)}{|a|^2 |b|^2 |b - a|^2} == \frac{2|a|^2 |b|^2 \Big(1 - \cos^2 \phi\Big)}{|a|^2 |b|^2 |b - a|^2} = \frac{2 \sin^2 \phi}{r_{23}^2}.$$

Here ϕ is the angle between a and b. The relation between the circumradius and the angle follows from the sine-theorem.

Let now x_i, x_j, x_k are three points in $\mathbb{R}^3$. As above r_{ij}, r_{ik}, r_{jk} are the distances between these points and R_{ijk} is the circumradius of the triangle obtained from these three points.
The value $1/R_{ijk}$ is called the Menger curvature (Karl Megner 1902-1985) of the triple (x_i, x_j, x_k) (coinsides with the usual curvature of the circle through these points).
If now

$$Q_N(x_1, \ldots, x_N) = \frac{\sum_{i \neq k, i, k \neq j}^{N} \frac{1}{R_{ijk}^2}}{2 \sum_{i \neq j}^{N} \frac{1}{r_{ij}^2}},$$

then

$$K(N) = \sup_{x_1, \ldots, x_N} Q_N(x_1, \ldots, x_N).$$

Lemma. *Let T be a triangle whose sides have the lengths a, b, c and whose circumradius is R. Then*

$$\frac{1}{R^2} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

with equality only for the equilateral triangle.

Proof. It is well known (for ex. see Mitrinovic:1989), that $a^2 + b^2 + c^2 \leq 9R^2$. The equality holds only for the equilateral triangle. Now combine this with the elemntary inequality

$$(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 9,$$

which follows immediately from $x/y + y/x \geq 2$, for $x, y > 0$.

Theorem. *Let* $u \in W^{1,2}(\mathbb{R}^{3N})$ *then*

$$\int_{\mathbb{R}^{3N}} |\nabla u|^2 dx \geq \frac{1}{2N} \int_{\mathbb{R}^{3N}} |u|^2 \sum_{i \neq j} \frac{1}{r_{ij}^2} dx.$$

Proof. It is enough to prove that $(4 + 4K(N))^{-1} \geq 1/2N$. By using the last lemma we find that

$$\sum_{i \neq k, i, k \neq j}^{N} \frac{1}{R_{ijk}^2} \leq \sum_{i \neq k, i, k \neq j}^{N} \left(\frac{1}{r_{ij}^2} + \frac{1}{r_{ik}^2} + \frac{1}{r_{jk}^2} \right) = (N-2) \sum_{i \neq j}^{N} \frac{1}{r_{ij}^2}.$$

Therefore

$$K(N) = \sup_{x_1, \ldots, x_N} \frac{\sum_{i \neq k, i, k \neq j}^{N} R_{ijk}^{-2}}{2 \sum_{i \neq j}^{N} r_{ij}^{-2}} \leq \frac{N-2}{2},$$

which proves our statement.

Remarks.

• This is already an improvement of the factor $(4N-4)^{-1}$ which we would have obtained by adding up standard 3D Hardy inequalities.

Open problem: Can one replace $1/2N$ by a constant independent of N as it has been shown for 1D N particles?

• For $N = 3$ and 4 the estimate $(4+4K(N))^{-1} \geq 1/2N$ is optimal. For larger N the value of $K(N)$ is unknown. Finding the sharp value of $K(N)$ is an interesting problem from geometrical combinatorics.

We have already proved that $K(N) \le (N-2)/2$.

Proposition. $\liminf_{N\to\infty} N^{-1} K(N) > 0$.

Proof. Let $R(x,y,z)$, $x,y,z \in \mathbb{R}^3$, now be the radius of the circumcircle of the triangle defined by (x,y,z) and let $r(x,y)$ be the distance between x and y. Assume that $\Omega \subset \mathbb{R}^3$ is an open bounded set with smooth boundary and let $\mathbb{Z}^3_M$ denote the three dimensional lattice $\{k/M,\, k \in \mathbb{Z}^3\}$. It is well known that $N := \#\{x_j \in \Omega \cap \mathbb{Z}^3_M\} \sim |\Omega| M^3 + o(M^3)$, as $M \to \infty$. Thus

$$\liminf_{N\to\infty} N^{-1} K(N) \\ \ge \lim_{N\to\infty} \frac{1}{|\Omega|} \frac{M^{-9} B_N}{M^{-6} A_N} = \frac{1}{2|\Omega|} \frac{\int_\Omega \int_\Omega \int_\Omega R^{-2}(x,y,z)\, dx dy dz}{\int_\Omega \int_\Omega r^{-2}(x,y)\, dx dy}.$$

If for example $\Omega = B = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ |x| < 1\}$, then by symmetry

$$\frac{1}{2}\int_{B^3} R^{-2}(x,y,z)\,dxdydz = \int_{B^3} \frac{(x_1 - z_1)(x_1 - y_1)}{|x-y|^2|x-z|^2}\ dxdydz$$
$$= \int_B \Big(\int_B \frac{x_1 - z_1}{|x-z|^2}\,dz\Big)^2\ dx > 0$$

and therefore

$$\liminf_{N\to\infty} N^{-1}K(N) \geq \frac{1}{|B|}\,\frac{\int_{B^3} \frac{(x_1-z_1)(x_1-y_1)}{|x-y|^2|x-z|^2}\ dxdydz}{\int_{B^2} r^{-2}(x,y)\,dxdy} > 0.$$

Proposition is proved.

• Suppose that the best asymptotic configuration of points could be described by a finite measure μ on $\mathbb{R}^3$. Then

$$\liminf_{N\to\infty} N^{-1} K(N) = \frac{1}{|\mu|} \frac{\int\int\int R^{-2}(x,y,z)\ d\mu(x)d\mu(y)d\mu(z)}{\int\int r^{-2}(x,y)\ d\mu(x)d\mu(y)},$$

The integral

$$\mathcal{C}(\mu) = \int\int\int R^{-2}(x,y,z)\ d\mu(x)d\mu(y)d\mu(z)$$

is known as Menger-Melnikov curvature of the measure μ.

Recently Tolsa used Menger-Melnikov curvature for geometric caracterisation of compact sets in the plane of zero analytic capacity (i.e. compact sets which are removable for bounded analytic functions). It has been a break through result prepared in the papers of David, Melnikov, Verdera, Mattila, Leger and many others.

Problem: Isoperimetric properties of the Menger-Melnikov curvature.

Find a measure μ, $\mu(\mathbb{R}^3) = 1$, such that the integral

$$I(\mu) = \frac{\int\int\int R^{-2}(x,y,z)\ d\mu(x)d\mu(y)d\mu(z)}{\int\int r^{-2}(x,y)\ d\mu(x)d\mu(y)}$$

is maximal.

Remark. Since $K(N) \leq \frac{N-2}{2}$ we have $I(\mu) \leq 1/2$.

2D case. Let

$$G(x) = \ln|x|, \qquad x = (x_1, x_2),$$

and let define

$$\mathbf{F} = -\frac{\nabla G}{G} = -\frac{1}{\ln|x|}\left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}\right).$$

Then $\operatorname{div}\mathbf{F} = \frac{1}{\ln^2|x|} \cdot \frac{1}{|x|^2}$ and applying the inequality

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} |u|^2 \Big(2\operatorname{div}\mathbf{F} - |\mathbf{F}|^2\Big)\,\mathrm{dx}$$

we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \ln^2|x|}\, dx.$$

2D multiparticle case.

We define

$$\mathbf{F} = \Big(\sum_{k \neq 1} \frac{1}{\ln |x_1 - x_k|} \frac{x_1 - x_k}{|x_1 - x_k|^2}, \ldots, \sum_{k \neq N} \frac{1}{\ln |x_N - x_k|} \frac{x_N - x_k}{|x_N - x_k|^2} \Big),$$

where $x_j \in \mathbb{R}^2$. Then the main lemma implies

Theorem.

$$\int_{\mathbb{R}^{2N}} |\nabla u|^2 dx \geq \frac{1}{4 + 4K(N)} \int_{\mathbb{R}^{2N}} |u|^2 \sum_{i \neq j} \frac{1}{\ln^2 r_{ij}} \frac{1}{r_{ij}^2} \, dx,$$

where as before $K(N) = \sup_{x_1, \ldots, x_N} \frac{\sum_{i \neq k, i,k \neq j}^N R_{ijk}^{-2}}{2 \sum_{i \neq j}^N r_{ij}^{-2}} \leq \frac{N-2}{2}$.

Remark. For 2D case we can only prove that for large N $c_1 N / \ln N \leq K(N) \leq N/2 - 1$.

2D magnetic Dirichlet Hardy inequality.

It has been shown in [LW] that if $\mathbf{F} = \alpha(-x_2\,|x|^{-2}, x_1\,|x|^{-2})$, $x = (x_1, x_2)$, $\alpha \in \mathbb{R}$, is an Aharonov-Bohm vector potential, then

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{F})u|^2\,dx \geq \min_{k\in\mathbb{Z}} (k-\alpha)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx.$$

Indeed, using polar coordinates (r,θ) we have $u(x) = \frac{1}{\sqrt{2\pi}} \sum_k u_k(r) e^{ik\theta}$. Therefore

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{F})u|^2\,dx = \int_0^\infty \int_0^{2\pi} \left(|u_r'|^2 + \left| \frac{iu_\theta' + \alpha u}{r} \right|^2 \right) r\,d\theta\,dr$$

$$\geq \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \left| \sum_k \frac{\alpha - k}{r} u_k e^{ik\theta} \right|^2 r\,d\theta\,dr = \int_0^\infty \sum_k \left| \frac{\alpha - k}{r} u_k \right|^2 r\,d\theta\,dr$$

$$\geq \min_{k\in\mathbb{Z}} (k-\alpha)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx.$$

Later Balinsky generalized this inequality and obtained a pretty result when $\mathbf{F}$ has a finite number of singularities by using conformal mappings.
Some interesting inequalities of this type were also recently obtained by Melgaard, Ouhabaz & Rozenblum.
Balinsky, Evans & Lewis used LW inequality for establishing a CLR inequalities in 2D case and later Bennewitz & Evans obtained LW type inequalities in L^p spaces.

Magnetic multiparticle case.

Let $x_j = (x_{j1}, x_{j2}) \in \mathbb{R}^2$, $j = 1, 2, \ldots, N$, and let

$$\mathbf{F_j} = \alpha\Big(-\sum_{k\neq j} \frac{x_{j2} - x_{k2}}{r_{jk}^2}, \sum_{k\neq j} \frac{x_{j1} - x_{k1}}{r_{jk}^2}\Big).$$

Theorem. *Let*

$$D_{N,\alpha} = \min_{l=1,\ldots,N-1} \Big(\frac{\min_{k\in\mathbb{Z}} |k - l\alpha|}{l}\Big)^2.$$

Then

$$\int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq D_{N,\alpha} \int_{\mathbb{R}^{2N}} |u|^2 \Big(\sum_{k\neq j} \frac{1}{r_{kj}^2}\Big) \, dx.$$

Proof.

Let $z = (z_1, \ldots, z_N)$, $z_j = x_{j1} + i x_{j2}$ and let $\Phi_j(z) = \Pi_{k \neq j}(z_j - z_k)$, $j, k = 1, \ldots, N$.

According to Balinsky's inequality there is a piecewise constant function $C_j(x) \geq D_{N,\alpha}$, such that

$$\int_{\mathbb{R}^{2N}} |i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq \int_{\mathbb{R}^{2N}} C_j(x) \left| \frac{(\Phi_j)'_{z_j}(z)}{\Phi_j(z)} \right|^2 |u|^2 \, dx.$$

Simple computation shows

$$\left| \frac{(\Phi_j)'_{z_j}(z)}{\Phi_j(z)} \right|^2 = \left| \sum_{k \neq j} \frac{1}{z_j - z_k} \right|^2 = \sum_{k,l \neq j} \frac{(x_j - x_k) \cdot (x_j - x_l)}{r_{jk}^2 r_{jl}^2}.$$

Therefore we obtain

$$\int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F_j})u|^2 \, dx \geq D_{N,\alpha} \int_{\mathbb{R}^{2N}} \sum_{j=1}^{N} \Big| \sum_{k\neq j}^{N} \frac{1}{z_j - z_k} \Big|^2 |u|^2 \, dx$$

$$= D_{N,\alpha} \int \Big(\sum_{k\neq j}^{N} \frac{1}{r_{jk}^2} + \sum_{l\neq k, l, k\neq j}^{N} \frac{1}{R_{jkl}^2} \Big) |u|^2 \, dx.$$

We complete the proof by noticing that

$$\min_{x\in\mathbb{R}^{2N}} \sum_{l\neq k, l, k\neq j}^{N} R_{jkl}^{-2} = 0.$$

A joke.

It is well known that the ground state of the harmonic oscillator

$$-\frac{d^2}{dx^2} + x^2 \quad \text{in} \quad L^2(\mathbb{R})$$

is $\varphi(x) = \exp(-x^2/2)$ and the corresponding eigenvalue $\lambda_1 = 1$. Therefore the following inequality is sharp

$$-\frac{d^2}{dx^2} + x^2 \geq 1.$$

Let us now consider the following formal expression:

$$A^*A = \Big(\frac{d}{dx} - x + \frac{1}{2x}\Big)\Big(-\frac{d}{dx} - x + \frac{1}{2x}\Big) \geq 0.$$

This can be rewritten as

$$A^*A = -\frac{d^2}{dx^2} + x^2 + \frac{1}{4x^2} - 1 - \frac{1}{2x^2} - 2x\,\frac{1}{2x} = -\frac{d^2}{dx^2} + x^2 - 2 - \frac{1}{4x^2}.$$

The latter identity implies

$$-\frac{d^2}{dx^2} + x^2 \geq 2 + \frac{1}{4x^2}.$$

Conclusion: The eigenvalue $\lambda_1 = 1$ does not exist, does it???

3D Coulomb case with N particles.

Theorem. *Let* $u \in W^{1,2}(\mathbb{R}^{3N})$. *Then*

$$\int_{\mathbb{R}^{3N}} |\nabla u|^2 dx - \int_{\mathbb{R}^{3N}} \Big(\sum_{i \neq j} \frac{1}{r_{ij}} \Big) |u|^2 \, dx \geq - \Big(\frac{N(N-1)}{2} + L(N) \Big) \int_{\mathbb{R}^{3N}} |u|^2 \, dx,$$

where

$$L(N) = \sup \sum_{j=1}^{N} \sum_{i \neq k, \, i,k \neq j} \frac{(x_j - x_i) \cdot (x_j - x_k)}{r_{ij} r_{jk}}.$$

Remarks.

• The sharp value of $L(N)$ is unknown except of $N = 3, 4, 5$. However, we can show that

$$\frac{1}{6}N(N-1)(N-2) \leq L(N) \leq \frac{1}{4}N(N-1)(N-2).$$

• The Coulomb case has also been recently studied by A.Mouchet who conjectured that for large N the optimal configuration is achieved when the N points are distributed uniformly on a sphere. In this case

$$0.0542... \leq \lim_{N\to\infty} N^{-3}L(N) \leq \frac{1}{18} = 0.0556...$$

Isoperimetric problem:

Let $x, y, z \in \mathbb{R}^3$ and let $c(x, y, z)$ be the the cosinus of the angle between the vectors $y - x$ and $z - x$.

Find a measure μ, $\mu(\mathbb{R}^3) = 1$, such that

$$J(\mu) = \int \int \int c(x, y, z)\, d\mu(x) d\mu(y) d\mu(z)$$

is maximal.

A question.

It would be interesting to obtain such type of inequalities for fermions.