Equality
of bulk and edge Hall conductances
in a mobility gap

(joint with A. Elgart and J. Schenker)

LMN

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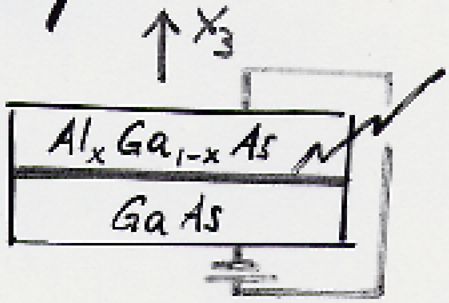
A typical sample

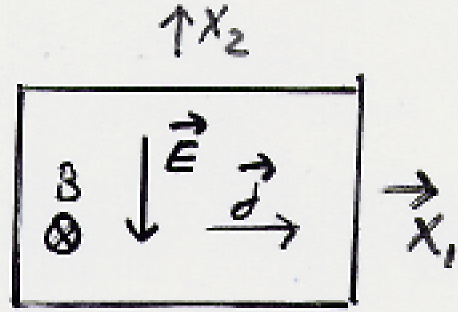
from the side:



electron gas

- confined to the interface (dim. = 2)
- of density n (or Fermi energy μ) tunable through gate potential!

from above:

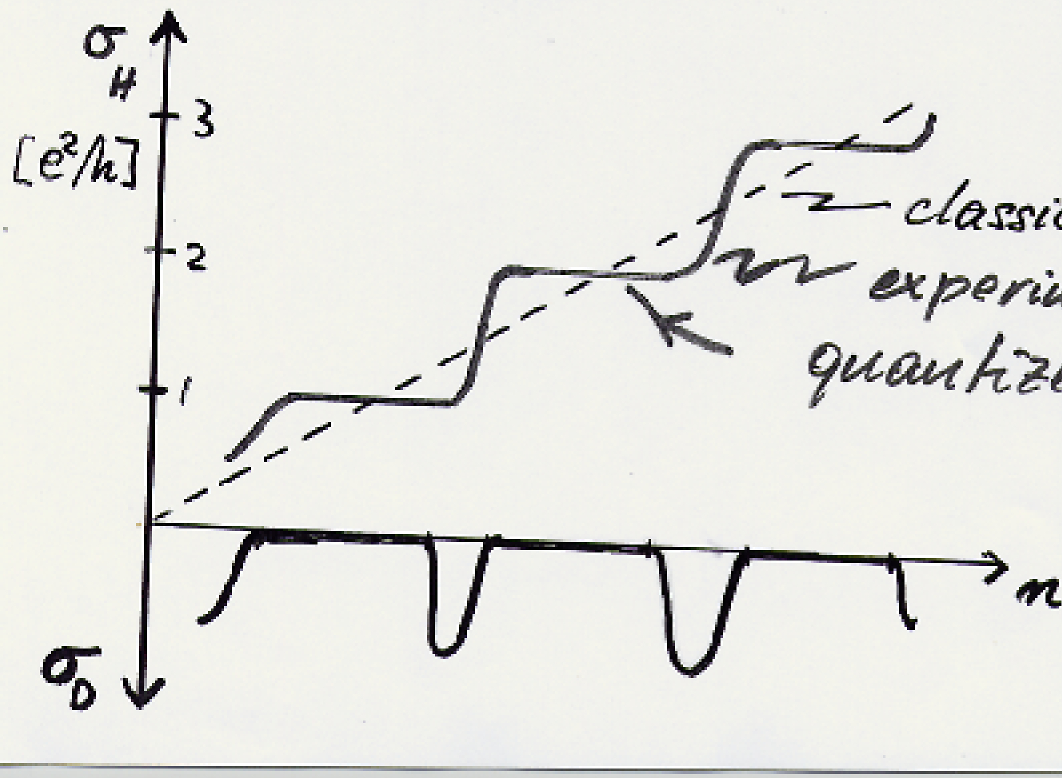


Hall-Ohm law

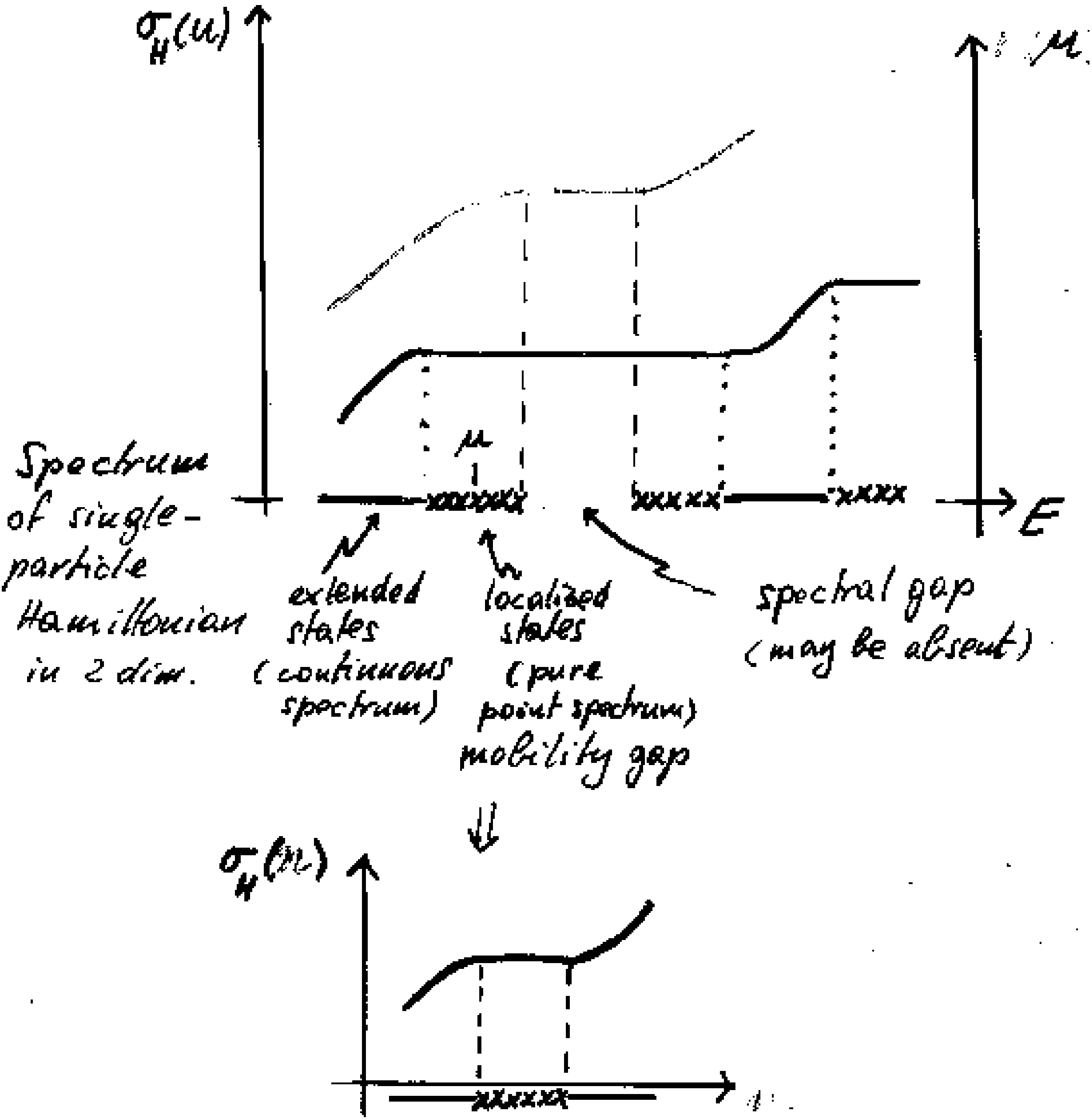
$$\vec{J} = \sigma \vec{E} \quad , \qquad \sigma = \begin{pmatrix} \sigma_D & -\sigma_H \\ \sigma_H & \sigma_D \end{pmatrix}$$

σ_H : Hall conductance

σ_D : ohmic (dissipative) conductance



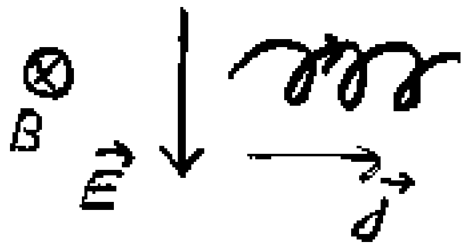
$\sigma_H(\mu)$



$\therefore$ plateaux arise because $\sigma_H(\mu)$ is constant for μ in a <u>mobility</u>, not a spectral, gap

Interpretations of the quantum Hall effect

1) as a bulk effect

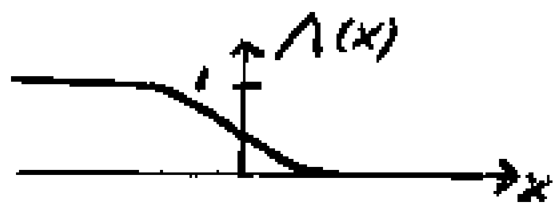


$$j_1 = -\sigma_H E_2$$

Kubo formula (linear response calculation)

$$\sigma_H \leftarrow \underset{\text{bulk}}{\sigma_B} := -i \operatorname{tr} P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]$$

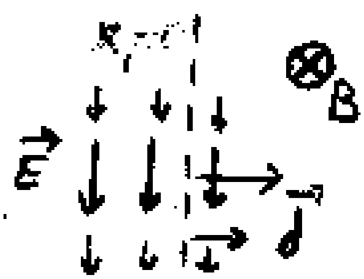
where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$



$$P_\mu = E_{(-\infty, \mu)}(H_B)$$

Fermi projection, i.e., onto occupied states of the Hamiltonian H_B (defined on the full plane)

- where from (sketch)?



$\longrightarrow$ electric current across $x_1 = 0$

$$I = \int_{-\infty}^{\infty} j_1 \, dx_2 \Big|_{x_1 = 0}$$

voltage

$$V = -\int_{-\infty}^{\infty} E_2 \, dx_2$$

current operator

$$-i[H_B, \Lambda_1]$$

$\longleftarrow$ Hall law

$$I = \sigma_H V$$

electric field

$$\vec{E} = \vec{\nabla} \Lambda_2$$

- Theorem (Bellissard, van Elst, Schulz-Baldes)

If μ is in a mobility gap, then $\sigma_D(\mu) = 0$ and $2\pi\,\sigma_B(\mu)$ is integer and constant.

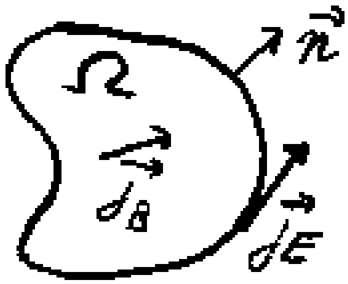
__Issue__

$$\sigma_B = \sigma_E$$

- Why care? (Halperin): a fraction of the current flows along the edge*, another in the bulk. Thanks to $\sigma_B = \sigma_E$ the quantization of σ_H is ensured.

(* in real samples: about 10 %)

- Why true? Non rigorous argument:



$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

rotation by $\pi/2$

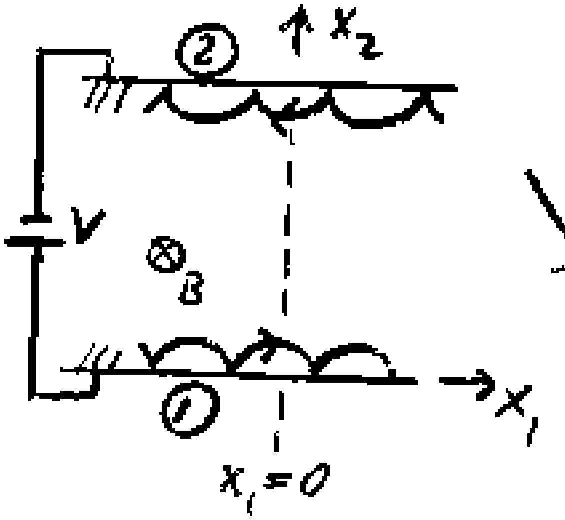
$$\vec{j}_B = \chi_\Omega \sigma_B \varepsilon \vec{E} \qquad\qquad \vec{j}_E = \sigma_E V \varepsilon \vec{n}\, \delta_{\partial\Omega}$$

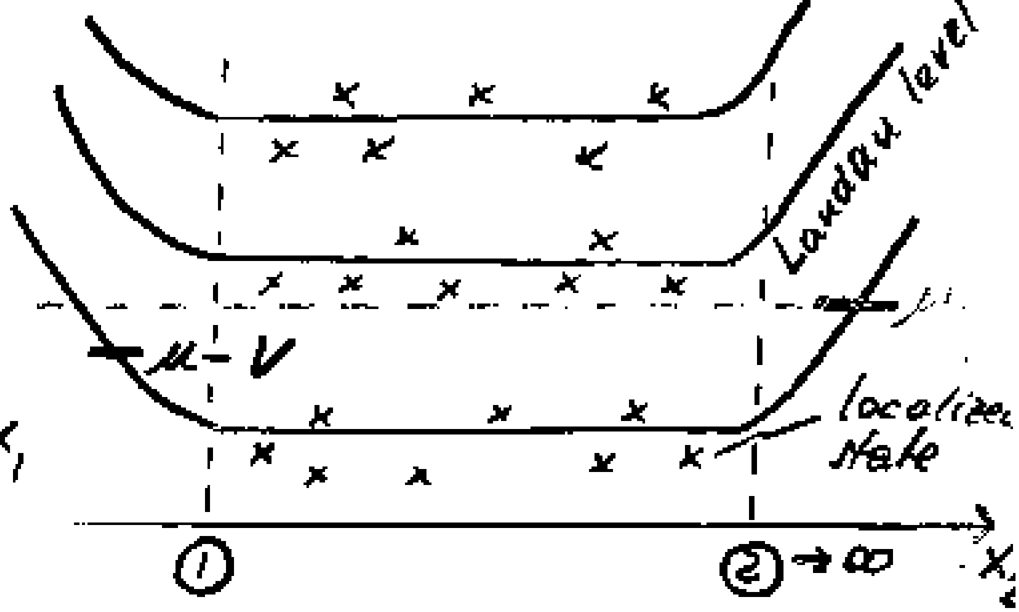
$$= -\chi_\Omega \sigma_B \varepsilon \vec{\nabla} V \qquad\qquad = -\sigma_E V \varepsilon \vec{\nabla} \chi_\Omega$$

$$\operatorname{div} \varepsilon \vec{v} = -\operatorname{curl} \vec{v}$$

$$\operatorname{div} \vec{j}_B = -\sigma_B \vec{\nabla}\chi_\Omega \cdot \varepsilon \vec{\nabla} V \qquad \operatorname{div} \vec{j}_E = \sigma_E \vec{\nabla} V \cdot \varepsilon \vec{\nabla} \chi_\Omega$$

$$\operatorname{div}(\vec{j}_B + \vec{j}_E) = 0 \quad \Rightarrow \quad \sigma_B = \sigma_E$$



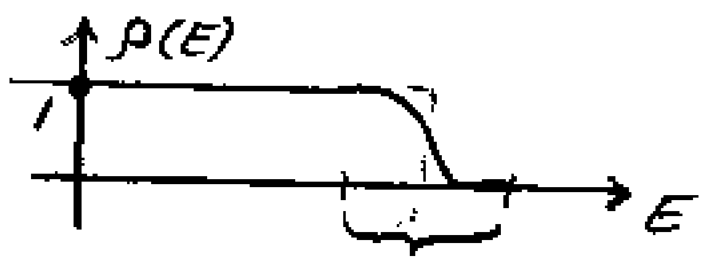


H_E : Hamiltonian on the upper half-plane (restriction of H_B, e.g. via Dirichlet bound. cond.

edge

$\rho(H_E)$: 1-particle density matrix, e.g.

$\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or smooth



Δ : gap for H_B (not H_E!)

If $V = 0$: no current across $x_1 = 0$

If $V \neq 0$:

$$I = \mathrm{tr}\left((\rho(H_E + V) - \rho(H_E))(-i)[H_E, \Lambda_1]\right)$$

As $V \to 0$: $I/V \to \sigma_H$

$$\sigma_H \sim \sigma_E := -i\,\mathrm{tr}\left(\rho'(H_E)[H_E, \Lambda_1]\right)$$

plane $\equiv$ lattice $\mathbb{Z}^2$

Hamiltonian H_B : operator on $\ell^2(\mathbb{Z}^2)$ with $H_B(x,x')$ of short range in $|x-x'|$ (tight binding)

- Theorem*. If Δ is a <u>spectral gap</u> for H_B, then

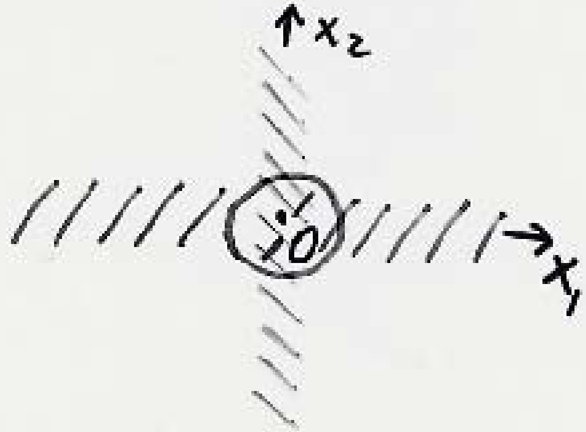
$$\sigma_B = \sigma_E$$

*Schulz-Baldes, Kellendonk and Richter ; later Elbau, G. ; Macris.

<u>Goal</u>: (A) spectral gap $\rightsquigarrow$ (B) mobility gap

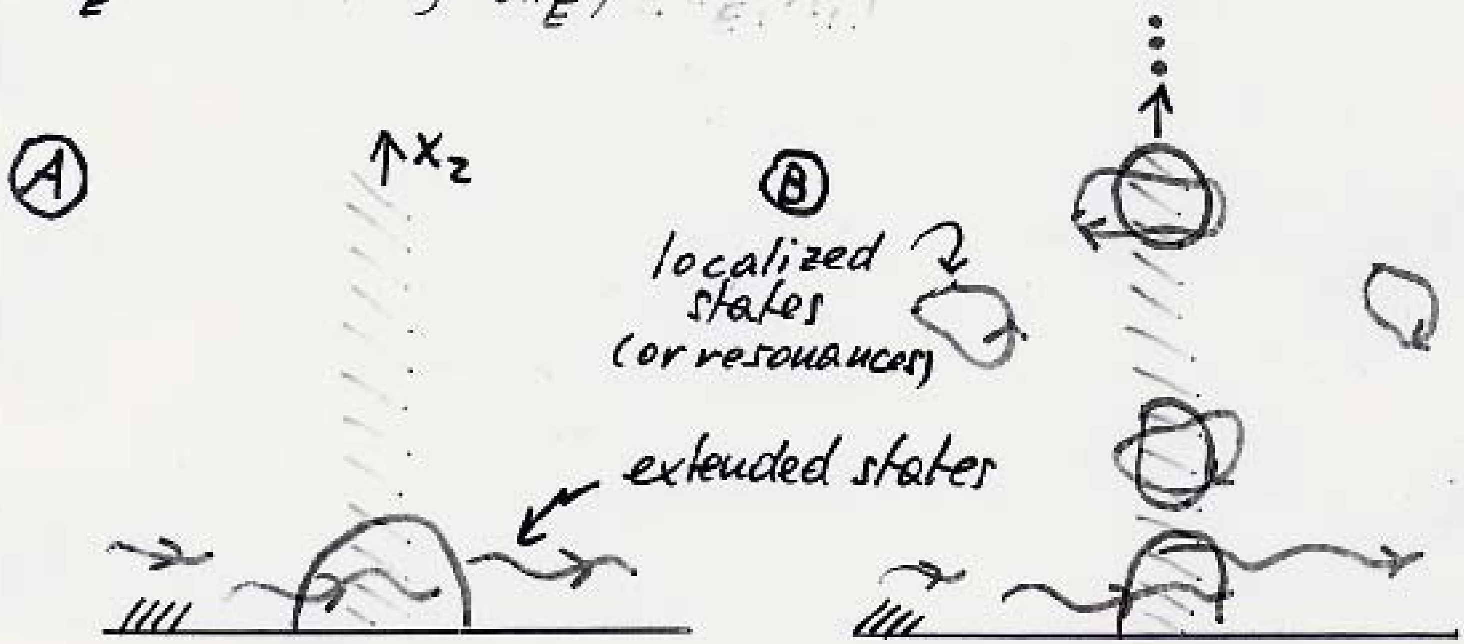
- Why are σ_B, σ_E well-defined?

Roughly: an operator has a well-defined <u>trace</u> if it acts non-trivially on a finite number of states only.



trace : yes

$$\sigma_E = -i\,\mathrm{tr}\, \rho'(H_E)[H_E, \Lambda_1]$$



trace : yes

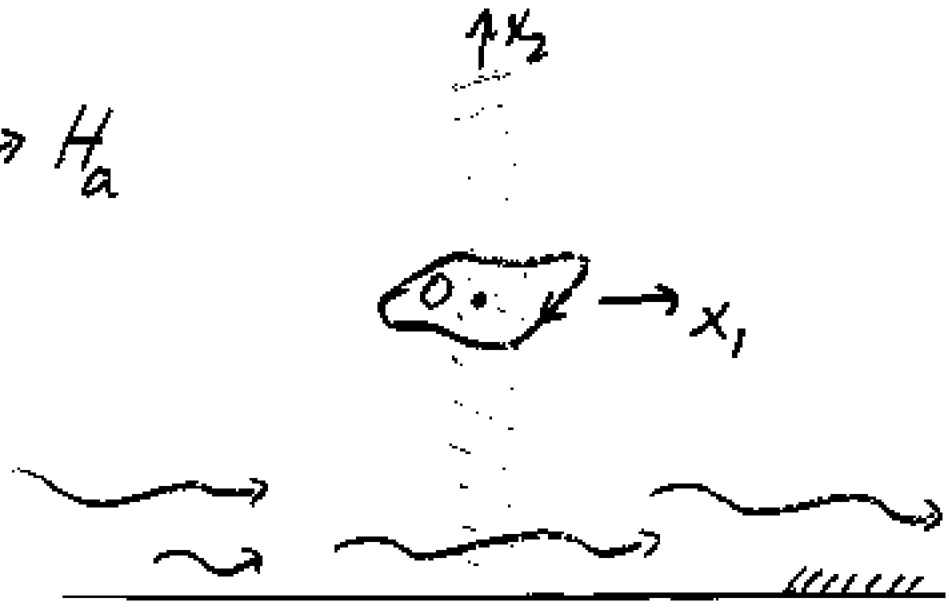
trace : no

$\therefore$ definition of σ_E in case B needs to be changed!

Localized states do not contribute to net current $\rightarrow$ sum them with care.

Definition of σ_E in case ③

$H_E \rightsquigarrow H_a$



edge $x_2 = -a$
eventually $-a \to -\infty$

current across the portion of $x_1 = 0$:

$$-i \operatorname{tr} \rho'(H_a)[H_a, \Lambda_1]\Lambda \qquad \text{(exists!)}$$

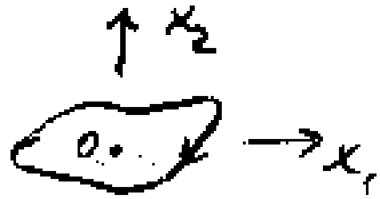
current across the portion : in the limit $a \to \infty$ pretend that the contributing states ψ_λ are localized ($H_B \psi_\lambda = \lambda \psi_\lambda$):

$$-\underbrace{i}_{\rho'(\lambda)}(\psi_\lambda, [H_B, \Lambda_1] \quad \psi_\lambda) = \underbrace{i}_{\rho'(\lambda)}(\psi_\lambda, [H_B, \Lambda_1] \quad \psi_\lambda)$$

Together:

$$\sigma_E^{(1)} := \lim_{a \to \infty} -i \operatorname{tr} \rho'(H_a)[H_a, \Lambda_1]\Lambda_2$$
$$+ i \sum_\lambda \rho'(\lambda)(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2 \psi_\lambda)$$

Idea: contributions from localized states to (lower) portion would cancel if x_2 in [illegible] were <u>time averaged</u>.



$$\sigma_E^{(2)} := \lim_{T\to\infty} \lim_{a\to\infty} -i \,\mathrm{tr}\, \rho'(H_a)\,[H_a, \Lambda_1]\, A_T(\Lambda_2)$$

where $A_T(\cdot)$ is the time average over $[0,T]$

$$A_T(\Lambda_2) = \frac{1}{T}\int_0^T e^{itH_a}\Lambda_2 e^{-itH_a}\,dt$$

- Theorem (Elgart, G., Schenker).

If $\operatorname{supp} \rho' \subset \Delta$ is a <u>mobility gap</u> for H_B then

•• the sums and the limits in $\sigma_E^{(1)}, \sigma_E^{(2)}$ exist

••
$$\sigma_E^{(1)} = \sigma_E^{(2)} = \sigma_B$$

•• In particular, $\sigma_E^{(1)}, \sigma_E^{(2)}$ do not depend on ρ', nor on boundary conditions

• Remark: Related result by Combes and Germinet for perturbations of Landau Hamiltonians.

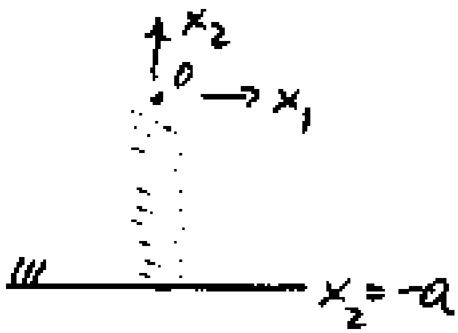
Reformulation of $\sigma_E = \sigma_B$.

$$\lim_{a\to\infty} -i \, tr \, \rho'(H_a)$$

$$= \sigma_B - i \sum_\lambda \rho'(\lambda) (\psi_\lambda, [H_B, \Lambda_1] \Lambda_2 \psi_\lambda)$$

Instead of proof, a simpler question :

Why does the limit exist ?



$$\underbrace{\| \rho'(H_a)[H_a, \Lambda_1] \Lambda_2 \|_1}_{\text{trace class norm}} \approx C \cdot a$$

We'll show

$$\rho'(H_a)[H_a, \Lambda_1] \Lambda_2 = Z(a) + \text{convergent remainder}$$

$$tr \, Z(a) = 0$$

Take

$$Z(a) = [\rho(H_a), \Lambda_1]\Lambda_2$$

$$Z(a)(x,x) = (\rho(H_a)(x,x)\Lambda_1(x) - \Lambda_1(x)\rho(H_a)(x,x))\Lambda_2(x)$$

$$= 0 \quad \to \quad \mathrm{tr}\, Z(a) = 0$$

Helffer–Sjöstrand representation

$$\rho(H_a) = \frac{1}{2\pi}\int_{\mathbb{C}} dz\ \partial_{\bar z}\rho(z)\, R_a(z), \qquad R_a(z) = (H_a - z)^{-1}$$

$$\rho'(H_a) = -\frac{1}{2\pi}\int dz\ \partial_{\bar z}\rho(z)\, R_a(z)^2$$

Thus

$$[\rho(H_a), \Lambda_1]\Lambda_2 = -\frac{1}{2\pi}\int dz\, \partial_{\bar z}\rho(z)\, R_a(z)[H_a, \Lambda_1]R_a(z)\Lambda_2$$

$$\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 = -\frac{1}{2\pi}\int dz\, \partial_{\bar z}\rho(z)\, \underbrace{R_a(z)^2[H_a, \Lambda_1]\Lambda_2}$$

$$= R_a(z)[H_a, \Lambda_1]\Lambda_2 R_a(z)$$

+ commutator (lump to Z(

Remainder

$$\frac{1}{2\pi}\int dz\, \partial_{\bar z}\rho(z)\, R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)$$

is convergent

as $a \to \infty$