Properads and homological differential operators related to surfaces

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Properads

DCor := Cor × Cor category of directed corollas

Properad $\mathcal{P}$ consists of

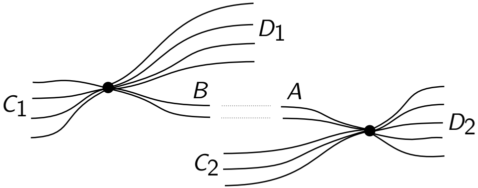
- collection $\{\mathcal{P}(C,D) \mid (C,D) \in \mathrm{DCor}\}$ of dg vector spaces
- two collections of degree 0 morphisms

$$\{\mathcal{P}(\rho,\sigma) : \mathcal{P}(C,D) \to \mathcal{P}(C',D') \mid (\rho,\sigma) : (C,D) \to (C',D')\}$$

$$\{ {}_B\overset{\eta}{\circ}_A \colon \mathcal{P}(C_1, D_1 \sqcup B) \otimes \mathcal{P}(C_2 \sqcup A, D_2) \to \mathcal{P}(C_1 \sqcup C_2, D_1 \sqcup D_2) \mid \eta : B \xrightarrow{\sim} A\}$$

satisfying the axioms:

1. Σ-bimodule
   $\mathcal{P}((1_C, 1_D)) = 1_{\mathcal{P}(C,D)}, \quad \mathcal{P}((\rho\rho', \sigma'\sigma)) = \mathcal{P}((\rho,\sigma))\,\mathcal{P}((\rho',\sigma'))$
2. equivariance
   $$(\mathcal{P}((\rho_1 \sqcup \rho_2|_{C_2}, \sigma_1|_{D_1} \sqcup \sigma_2))\ {}_B\overset{\eta}{\circ}_A = {}_{\sigma_1(B)}\overset{\rho_2\eta\sigma_1^{-1}}{\circ}_{\rho_2(A)} (\mathcal{P}((\rho_1,\sigma_1)) \otimes \mathcal{P}((\rho_2,\sigma_2))$$
3. associativity ...



Additional grading by $\mathbb{N}_0$ - **genus** G or by the **Euler characteristic** χ

$$\chi = 2G + |C| + |D| - 2$$

$\Longrightarrow$ components $\mathcal{P}(C, D, \chi)$
We assume only **stable components**, i.e. $\chi > 0$

Example: (Closed) **Frobenius properad** $\mathcal{F}$:

$\mathcal{F}(C, D, \chi) = \mathbb{k}$
$\Longrightarrow$ has trivial differential and Σ-structure
$\Longrightarrow$ ${}_B\overset{\eta}{\circ}_A$ do not depend on sets A, B

Geometrically: 2-dim compact oriented surfaces with punctures in the interior,
$g = g_1 + g_2 + |A| - 1$

Example: **Endomorphism properad $\mathcal{E}_V$**:

For (V,d) dg vector space, $(C,D) \in \mathrm{DCor}$, $\chi > 0$ define

$$\mathcal{E}_V(C,D,\chi) := \mathrm{Hom}_{\Bbbk}\Big(\bigodot_D V, \bigodot_C V\Big)$$

For $\bar{f} \in \mathrm{Hom}_{\Bbbk}(\bigotimes_D V, \bigotimes_C)$ corresponding to $f \in \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V)$

$$d(\bar{f}) = \sum_{i=0}^{m-1} (1^{\otimes i} \otimes d \otimes 1^{\otimes m-i-1})\bar{f} - (-1)^{|\bar{f}|} \sum_{i=0}^{n-1} \bar{f}(1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1})$$

Algebra over properad is a properad morphism $\alpha : \mathcal{P} \to \mathcal{E}_V$, i.e.

$$\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi) \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathrm{DCor}, \chi > 0\}$$

satisfying: $\alpha \circ \mathcal{P}(\rho,\sigma) = \mathcal{E}_V(\rho,\sigma) \circ \alpha$

$$\alpha \circ ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{P}} = ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha)$$

Cobar complex

Directed graph G

Assign a non-negative integer $G := \dim_{\mathbb{Q}} H_1(\mathrm{G}, \mathbb{Q}) + \sum_i G_i$

The stable graph satisfies for every vertex V_i

$$2(G_i - 1) + |C_i| + |D_i| > 0$$

Cobar complex of properad $\mathcal{P}$

- Elements are iso class of G with "decoration" by element

$$(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n)$$

$$\partial_{C\mathcal{P}} = d_{P\#} \otimes 1 + \sum_{\substack{C_1 \sqcup C_2 = C \\ D_1 \sqcup D_2 = D \\ \chi_1, \chi_2 > 0 \\ \chi}} \frac{1}{|A|!} \left({}^{(C_1, D_1 \sqcup B, \chi_1)} \, {}_{B}\overset{\eta}{\circ}{}_{A} \, {}^{(C_2 \sqcup A, D_2, \chi_2)} \right)_P^{\#} \otimes (\uparrow V \wedge \cdot)$$

Theorem: Algebra over the cobar complex $C\mathcal{P}$

Algebra over $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection $\left\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi)^{\#} \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathsf{DCor}, \chi > 0\right\}$ of deg 1 linear maps s.t.

$$\mathcal{E}_V(\rho,\sigma) \circ \alpha(C,D,\chi) = \alpha(C',D',\chi) \circ \mathcal{P}(\rho^{-1},\sigma^{-1})^{\#}$$

$$d \circ \alpha = \alpha \circ d_{\mathcal{P}^{\#}} + \sum \frac{1}{|A|!} ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha) \circ ({}_B\overset{\eta}{\circ}{}_A)^{\#}_{\mathcal{P}}$$

By isomorphism

$$\mathrm{Hom}_{\Sigma_C \times \Sigma_D}(\mathcal{P}(C,D,\chi)^{\#}, \mathcal{E}_V(C,D,\chi)) \xrightarrow{\cong} {}^{\Sigma_C}(\mathcal{P}(C,D,\chi) \otimes \mathcal{E}_V(C,D,\chi))^{\Sigma_D}$$

we can rewrite algebra over $C\mathcal{P}$ as element

$$L \in \prod_{\substack{|C|,|D| \\ \chi > 0}} {}^{\Sigma_C}(\mathcal{P}(C,D,\chi) \otimes \mathcal{E}_V(C,D,\chi))^{\Sigma_D}$$

satisfying **Master equation** $d(L) + L \circ L = 0$ with differential

$$d = d_{\mathcal{P}} \otimes 1_{\mathcal{E}_V} - 1_{\mathcal{P}} \otimes d_{\mathcal{E}_V}$$

The invariants are isomorphic to coinvariants so we get an isomorphism

$$\prod_{\substack{|C|,|D|\\ \chi>0}} {}^{\Sigma_C}\left(\mathcal{P}(C, D, \chi) \otimes \mathcal{E}_V(C, [n], \chi)\right)^{\Sigma_n} \cong$$

$$\prod_{\substack{|C|,|D|\\ \chi>0}} \left(\mathcal{P}(C, D, \chi)_{\Sigma_C} \otimes_{\Sigma_D} (V^{\otimes C} \otimes (V^{\#})^{\otimes D}\right)$$

with "transfered" differential and composition maps

If $|C|, |D| \geq 1$ we can introduce positional derivations

For simplicity assume $C = \{1, \dots m\}, D = \{1, \dots n\}$

$$\frac{\partial^{(k)}}{\partial a_j} \left(a_{i_1} \dots a_{i_{m_2}} \right) = (-1)^{|a_j|(|a_{i_1}| + \dots |a_{i_{k-1}}|)} \delta_j^{i_k} \left(a_{i_1} \dots \widehat{a_{i_k}} \dots a_{i_{m_2}} \right)$$

for sets $J = \{j_1, \dots j_{|N|}\}$ and $K = \{k_1, \dots k_{|N|}\}$

$$\frac{\partial^{(K)}}{\partial a_J} = \frac{\partial^{(k_1)}}{\partial a_{j_1}} \dots \frac{\partial^{(k_{|N|})}}{\partial a_{j_{|N|}}}.$$

And interpret the "inputs" as partial derivations acting on "outputs"