Operads for algebraic quantum field theory

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Higher Structures in M-Theory, 14-08-2018

Based on joint work with Alexander Schenkel

Algebraic quantum field theory

Input categories

- $\mathbf{Loc}$: globally hyperbolic Lorentzian spacetimes
- $\mathbf{Alg}$: associative algebras

def An AQFT is a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}$ satisfying:

- Einstein causality: if $M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2$ are spacelike separated embeddings,

$$\Big[\mathfrak{A}(f_1)(\mathfrak{A}(M_1)), \mathfrak{A}(f_2)(\mathfrak{A}(M_2))\Big] = \{0\}$$

- time-slice axiom: if $f(M) \subseteq N$ contains a Cauchy surface of N,

$$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\cong} \mathfrak{A}(N)$$

rem Analogous structures for:

- classical field theories (with Poisson algebras)
- linear field theories (with Heisenberg Lie algebras from presymplectic vector spaces)

Operadic approach

General input data:

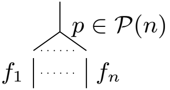
- $\mathbf{C}$: small category of spacetimes
- $\mathcal{P}$: uncoloured operad

Interested in functors $\mathfrak{A} : \mathbf{C} \to \mathbf{Alg}(\mathcal{P})$ satisfying a suitable generalization of Einstein causality (generalizing [Benini-Schenkel-Woike, '17])

rem Time-slice is implemented via localization of categories $\mathbf{C} \to \mathbf{C}[W^{-1}]$

def The $\mathbf{C}$-colouring of $\mathcal{P}$ is the $\mathrm{Ob}(\mathbf{C})$-coloured operad

$$\mathcal{P}_{\mathbf{C}}\big((c_1, \dots, c_n), t\big) = \coprod_{\underline{f} : (c_1, \dots, c_n) \to t} \mathcal{P}(n) \ = \Big\{ \quad \Big\}$$



lem $\mathbf{Alg}(\mathcal{P}_{\mathbf{C}}) \cong \mathrm{Fun}\big(\mathbf{C}, \mathbf{Alg}(\mathcal{P})\big)$

? How do we implement Einstein causality?

Operadic approach to Einstein causality

Extra input:

- $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$: spacetime category $\mathbf{C}$ with orthogonality relations $\perp$
  i.e. pairs of maps $c_1 \xrightarrow{f_1} t \xleftarrow{f_2} c_2$ with the same target
- $\mathcal{P}^r$: operad $\mathcal{P}$ with a double pointing $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \mathcal{P}$

def The field theory operad of type $\mathcal{P}^r$ on $\overline{\mathbf{C}}$ is the coequalizer

$$\perp \otimes R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \mathcal{P}_{\mathbf{C}} \dashrightarrow \mathcal{P}^r_{\overline{\mathbf{C}}}$$

thm $\mathbf{Alg}\big(\mathcal{P}^r_{\overline{\mathbf{C}}}\big) \cong \Big\{$Functors $\mathbf{C} \to \mathbf{Alg}(\mathcal{P})$ satisfying Einstein causality$\Big\}$.

rem The construction of $\mathcal{P}^r_{\overline{\mathbf{C}}}$ is functorial in $\mathcal{P}^r$ and $\overline{\mathbf{C}}$,
allowing for comparisons between different types of field theories

Example: linear quantization adjunction

General result: an operad map $\mathcal{O} \xrightarrow{\phi} \mathcal{Q}$ induces an adjunction

$$\phi_! : \mathbf{Alg}(\mathcal{O}) \leftrightarrows \mathbf{Alg}(\mathcal{Q}) : \phi^*$$

ex $u\mathcal{L}ie \to \mathcal{A}s$, from $[\ ,\ ] \mapsto \mu - \mu^{op}$, defines a linear quantization adjunction

$$Q : \underbrace{\text{Linear field theories}}_{\mathbf{Alg}\left(u\mathcal{L}ie_{\overline{\mathbf{C}}}^{[,]=0}\right)} \rightleftarrows \underbrace{\text{Quantum field theories}}_{\mathbf{Alg}\left(\mathcal{A}s_{\overline{\mathbf{C}}}^{\mu=\mu^{op}}\right)} : U$$

Properties:

- For a functor to presymplectic vector spaces $\mathfrak{L} : \mathbf{C} \to \mathbf{Symp}$, $Q(\mathsf{Heis} \circ \mathfrak{L}) = \mathsf{CCR}(\mathfrak{L})$, the traditional canonical quantization of $\mathfrak{L}$
- Q preserves the time-slice axiom and descent properties in field theories

why For chain complex-valued field theories this is a Quillen adjunction
$\Rightarrow$ derived quanization of linear gauge theories