Gauge PDE, AKSZ sigma models, and higher spin theories

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Motivation

- Theories of fundamental interactions (Gravity, YM, Strings, M-Theory, Higher-spin theories . . .) are inevitably gauge theories.

- Batalin-(Fradkin-) Vilkovisky (BV/BFV) approach (and its generalizations) gives a proper language for gauge gauge theories. *Batalin, (Fradkin), Vilkovisky, 1981 . . .* . It is the framework in which L_∞ originally appeared in physics.

- The relation with Q-manifolds becomes manifest in the *Alexandrov-Kontsevich-Schwartz-Zaboronsky 1994* (AKSZ) form of BV for topological theories. In so doing the equations of motion, gauge symmetries, etc. are encoded in a homological vector field Q on the target space.

- AKSZ formulation has certain advantages over the usual jet-space version of the BV formalism *Henneaux; Barnich, Brandt, Henneaux*:including manifest background independence of AKSZ useful in studying boundary values, manifest realization of symmetries etc., close to unfolded formalism *Vasiliev 1988., . . .* of higher spin (HS) theories.

- Applications: higher spin gauge theories *Fronsdal, Fradkin-Vasiliev, Fradkin-Tseytlin, Vasiliev, . . .* HS theories are interesting on their own as nontrivial extensions of gravity, which share background independence, relation to geometry, etc.

- Anti de Sitter (AdS) HS theories - holographic duals of simple CFTs (free scalar)

- We lack first-principle understanding/derivation of *Vasiliev (1991)* system

Plan

- Jet-bundles and PDE
- Gauge PDE
- AKSZ sigma models
- Parent formulation
- Higher spin (HS) fields
- HS ineraction and holography

PDEs and jet-bundles

Fiber-bundle $\mathcal{F} \to X$ (global aspects are not discussed):
base space (independent variables or space-time coordinates): x^a, $a = 1, \ldots, n$.
Fiber: (dependent variables or fields ϕ^i)

Jet-bundle:
A point of J^n is a pair $(x, [s])$, where $[s]$ is an equivalence class of sections $s : X \to \mathcal{F}$ such that their partial derivatives at x coincide to order n. In coordinates:

$$\frac{\partial^l \phi^i(s(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} = \frac{\partial^l \phi^i(s'(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} \qquad l = 0, 1, \ldots n$$

In particular, $J^0(\mathcal{F}) = \mathcal{F}$.

One can use x^i, and values of above derivatives as coordinates:

$J^0(\mathcal{F}):\quad x^a, \phi^i, \quad J^1(\mathcal{F}):\quad x^a, \phi^i, \phi^i_a\,, \quad J^2(\mathcal{F})\quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}\,, \quad \ldots$

Projections:

$$\ldots \to J^N(\mathcal{F}) \to J^{N-1}(\mathcal{F}) \to \ldots \to J^1(\mathcal{F}) \to J^0(\mathcal{F}) = \mathcal{F}$$

Useful to work with $\mathcal{J} := \mathcal{J}^\infty$ (projective limit).

A local function is a pull-back of a function from $J^N(\mathcal{F})$ for some N. i.e. it depends on only a finite number of the coordinates.
A local function $f = f(x, \phi, \phi_a, \phi_{ab} \ldots)$ can be evaluated on a section $s : X \to \mathcal{F}$ as

$$f(s) := f(x, \phi^i(s), \partial_a \phi^i(s), \ldots)$$

Total derivative: (imitates the action of standard partial derivative)

$$\partial_a^T := \frac{\partial}{\partial x^a} + \phi_a^i \frac{\partial}{\partial \phi^i} + \phi_{ab}^i \frac{\partial}{\partial \phi_a^i} + \dots$$

Main property:

$$\partial_a(f(s)) = (\partial_a^T f)(s)\,.$$

Similarly one defines local forms. These are forms that can be obtained by pullback from finite jets.

Space-time differentials dx^a. Horizontal differential:

$$d_{\mathsf{h}} \equiv dx^a \partial_a^T \,, \qquad d_{\mathsf{h}}^2 = 0 \,.$$

Differential forms:

$$\alpha = \alpha(x, dx, \phi, \phi_a, \ldots)_{I_1 \ldots I_k} d_{\mathsf{v}}\phi^{I_1} \ldots d_{\mathsf{v}}\phi^{I_k} \,, \qquad \phi^I = \{\phi^i_{a_1 \ldots a_m}\}$$

Vertical differential:

$$d_{\mathsf{v}} \equiv d - d_{\mathsf{h}} = d_{\mathsf{v}}\phi^I \frac{\partial}{\partial \phi^I}$$

Variational bicomplex:

$$d_{\mathsf{v}}^2 = 0, \qquad d_{\mathsf{v}} d_{\mathsf{h}} + d_{\mathsf{h}} d_{\mathsf{v}} = 0 \,, \qquad d_{\mathsf{h}}^2 = 0$$

Bidegree (l, p). Locally on the jet space $H^{(0,k)}(d_{\mathsf{v}}) = 0 = H^{(<n,0)}(d_{\mathsf{h}})$, where $n = \dim(X)$. $H^{(n,0)}(d_{\mathsf{h}}) =$ local functionals

A system of partial differential equations (PDE) is a collection of local functions on $\mathcal{J}$

$$E_\mu[\phi, x] \,.$$

The equation manifold (stationary surface): $E \subset \mathcal{J}$ singled out by: (prolonged equation)

$$\partial^T_{a_1} \ldots \partial^T_{a_l} E_\mu = 0 \,, \qquad l = 0, 1, 2, \ldots$$

understood as the algebraic equations in $\mathcal{J}$.

∂^T_a are tangent to E and hence restricts to E. So do the differentials d_{h} and d_{v}. $\partial^T_a|_E$ determine a dim-n involutive distribution – Cartan distribution.

Definition: *[Vinogradov] PDE is a manifold E equipped with a Cartan distribution $C(E) \subset TE$.*

In addition one typically assumes regularity, constant dimension, and E is a bundle over the spacetime.

PDEs are isomorphic when the respective distributions are.

Differential forms on E form the variational bicomplex of E. Note that in general $H^k(d_{\mathsf{h}}) \neq 0$ for $k < n$.

For $n = 0$ PDEs are just usual manifolds.

Use the supergeometry language and define $\mathcal{E} := C[1]E$ so that the equation is a Q-manifold $(\mathcal{E}, d_{\mathsf{h}})$. $C^\infty(\mathcal{E})$ are horizontal forms.

Example: mechanics

ODE: $x_{tt} = f(x, x_t, t)$, as cooridnates on $\mathcal{E}$ one can take t, dt, x, x_t i.e. $\mathcal{E}$ is a phase space extended by time variable. If the equation arise from $L(x, x_t, t)$ then $\mathcal{E}$ acquires (pre)symplectic (and contact) structure.

Example: scalar field

Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)\,, \qquad \partial_a\partial^a\phi + \frac{\partial V}{\partial \phi} = 0\,.$$

$\mathcal{E}$ is coordinatized by $x^a, dx^a, \phi, \phi_a, \phi_{ab}, \ldots$. Already ϕ_{ab} are not independent. One can e.g. take $\phi_{abc\ldots}$ traceless. The d_{h}-differential on $\mathcal{E}$ reads as

$$d_{\mathsf{h}} x^a = dx^a\,, \quad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \quad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})\,, \quad \ldots$$

So if the system is nonlinear, i.e. $\frac{\partial V}{\partial \phi}$ nonlinear in ϕ, d_{h} is also nonlinear.

Linear PDE:

$$E_\mu = D_{\mu i}(x, \partial_a^T)\phi^i$$

Equation manifold is a vector bundle over X.
Formal solutions at $x_0 \in X$:

$$\phi^i(x_0, y) = \phi^i(x_0) + \partial_a \phi^i y^a + \frac{1}{2}\partial_a\partial_b\phi^i(x_0)y^a y^b + \dots$$

$$W_{x_0} = \{\phi^i(x_0, y) \;:\; D_{\alpha i}(x_0 + y, \frac{\partial}{\partial y})\phi^i(x_0, y) = 0\}$$

give a fiber at x_0.

Strictly speaking, in general it's not a vector bundle. Often in applications: there is a transitive space-time symmetry which forces all fibers to be isomorphic.

Moreover, fiber is a module over the space-time symmetry algebra. Sometimes called Weyl module in HS context *(cf. talk by Vasiliev).*

Gauge PDE

Definition: Q-manifold (M, Q, gh) is a graded supermanifold M equipped with the odd nilpotent vector field of degree 1, i.e.

$$Q^2 = 0\,, \qquad |Q| = 1\,, \quad \mathrm{gh}(Q) = 1$$

Example: Odd tangent bundle: $(T[1]M, d)$. If ξ^i are coordinates on the fibres of $T[1]M$ in the basis $\frac{\partial}{\partial z^i}$:

$$d := \xi^i \frac{\partial}{\partial z^i}$$

Example: CE complex $(\mathfrak{g}[1], Q_{CE})$. If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}[1]$ is equipped with Q structure. If c^α are coordinates on $\mathfrak{g}[1]$ in the basis e_α then

$$Q_{CE} f = c^\alpha c^\beta U^k_{\alpha\beta} \gamma \frac{\partial}{\partial c^\gamma}\,, \qquad [e_\alpha, e_\beta] = U^\gamma_{\alpha\beta} e_\gamma$$

Example: $(V[1](M), Q)$ where $V(M)$ Lie algebroid. Indeed generic Q of degree 1 locally reads

$$Q = c^\alpha R_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta}(z) \frac{\partial}{\partial c^\gamma}$$

R_α determines anchor, $U^\gamma_{\alpha\beta}$ bracket on sections, Q^2 encodes compatibility.

Gauge PDE in $n = 0$ (trivial Cartan distribution) is a nonnegatively graded Q-manifold $(\mathcal{E}, Q)$.

If only ghost degree $0, 1$ variabels are present then it is just a Lie algebroid.

Proposition: [AKSZ, 1994] Let $p \in \mathcal{E}$ and $Q|_p = 0$ then $T_p\mathcal{E}$ is an L_∞ algebra.

Important feature: although this is an intrinsic definition ($\mathcal{E}$ is not embedded into some "jet space") there are infinitely many Q-manifolds representing the same gauge PDE.
Example: Given (M, Q) take (Q', M') as follows:

$$M' = M \times T[1]\mathbb{R}^k\,, \qquad Q' = Q + d_{T[1]\mathbb{R}^k}$$

It is clear that these are homotopy equivalent Q-manifolds.
In the context of gauge theories coordinates on $T[1]\mathbb{R}^k$ are known as "generalized auxiliary fields" *Henneaux, 1990* (in the Lagrangian setting). In general homotopy equivalence.

Often one can find "minimal" Q-manifold describing a given equation. In simple case it's a direct analog of "minimal model" in L_∞ algebras.

Batalin–Vilkovisky formalism

If the theory is Lagrangian then:

$E_i = \frac{\delta S_0}{\delta \phi^i}$, reducibility relations/gauge generators $R^i_\alpha E_i = 0$

Natural bracket structure (antibracket)

$$\left(\phi^i, \phi^*_j\right) = \delta^i_j \qquad \left(c^\alpha, \mathcal{P}_\beta\right) = \delta^\alpha_\beta$$

BV master action

$$s = \left(\cdot, S_{BV}\right), \qquad S_{BV} = S_0 + \phi^*_i R^i_\alpha c^\alpha + \ldots$$

Master equation:

$$\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0$$

Gauge PDE $n \geqslant 0$

If PDE $(\mathcal{E}_0, d_\mathsf{h})$ has gauge symmetries there are parameters ϵ^α which are arbitary space time functions. Promote them to ghost variables c^α and consider the extension $\mathcal{E}$ of $\mathcal{E}_0$ by the jet-space for c^α:

$$C^I = \{c^\alpha, \quad c^\alpha_a, \quad c^\alpha_{ab}, \quad \ldots\}$$

The gauge symmetry is encoded in vector field γ satisfying

$$[d_\mathsf{h}, \gamma] = 0\,, \qquad \gamma^2 = 0\,, \qquad \mathsf{gh}(\gamma) = 1$$

It can be written as

$$\gamma = C^I R^A_I(\psi) \frac{\partial}{\partial \psi^A} - \frac{1}{2} C^I C^J U^K_{IJ}(\psi) \frac{\partial}{\partial C^K}$$

Vector fields R_I determine an involutive distribution on $\mathcal{E}_0$ (gauge-distribution), compatible (in the sense of $[d_\mathsf{h}, \gamma] = 0$) with Cartan distribution.

The above motivates the following (somewhat provisional) definition:

Definition: gauge PDE $(\mathcal{E}, s, d_{\mathsf{h}})$ is a Q-manifold $(\mathcal{E}, s)$ equipped with Cartan distribution d_{h} compatible with s and such that $H^i(s) = 0$ for $i < 0$.

The compatibility condition reads as:

$$[d_{\mathsf{h}}, s] = 0$$

It turns out that any local gauge theory gives rise to $(\mathcal{E}, s, d_h)$ (e.g. just by constructing BV formulation). Other way around, given $(\mathcal{E}, s, d_{\mathsf{h}})$ one can systematically reconstruct certain explicit realization of this system.

Subtleties:

- generic $(\mathcal{E}, s, d_{\mathsf{h}})$ may give intractable theory with infinite amount of fields and/or derivatives of unbounded order
- in contrast to usual PDE one and the same gauge PDE can be described by many equivalent $(\mathcal{E}, s, d_{\mathsf{h}})$. Possible way out is to ask for "minimal" $(\mathcal{E}, s, d_{\mathsf{h}})$

Linear Gauge PDE

Work in terms of $\mathcal{J}$ – jet-bundle extended by ghosts and anti-fields. Gauge PDE: $(\mathcal{J}, s, d_{\mathsf{h}})$
$(s = \delta + \gamma + \ldots)$

Linear s (i.e. a linear pieace in the expansion of s around a "vacuum solution")

$$s\psi^A = \Omega^A_B(x, \partial^T_a)\psi^B\,, \qquad s^2 = 0 \quad \rightarrow \quad \Omega^A_B \Omega^B_C = 0$$

Introduce a graded vector bundle $\mathcal{H}(X)$ over X underlying the space of fields, ghosts, etc. Sections:

$$\Phi = \phi^A(x) e_A\,, \quad \deg(e_A) = -\mathrm{gh}(\psi^A)$$

"First quantized BRST complex"

Decompose $\mathcal{H} = \bigoplus \mathcal{H}_l$ and $\Phi = \sum \Phi^{(l)}$ accordingly.
As $\deg \Omega = 1$, $\Omega^2 = 0$ we have complex: *cf. talk by Hohm*

$$\dots \xrightarrow{\Omega^{(-2)}} \Gamma(\mathcal{H}_{-1}) \xrightarrow{\Omega^{(-1)}} \Gamma(\mathcal{H}_0) \xrightarrow{\Omega^{(0)}} \Gamma(\mathcal{H}_1) \xrightarrow{\Omega^{(1)}} \dots$$

Equations of motion and gauge symmetries:

$$\Omega^{(0)}\Phi^{(0)} = 0\,, \qquad \delta\Phi^{(0)} = \Omega^{(-1)}\Phi^{(-1)}\,, \qquad \dots$$

Has a clear interpretation as a (formal) quantum mechanics of a constrained system.

If in addition $\Gamma(\mathcal{H})$ is equipped with degree -1 inner product $\int d^n x \langle \cdot, \cdot \rangle$ such that Ω is formally selfadjoint:

$$S_{BV} = \int d^n x \langle \Psi, \Omega\Psi \rangle\,, \qquad \Psi = \psi^A e_A \quad \text{– string field}$$

(cf. quadratic SFT action) *Bochicchio, Thorn 1986*

Gauge ODE. BFV formalism

Typically, for $n = 1$ $\mathcal{E}$ is can be taken finite-dimensional. For simplicity: $\mathcal{E}$ is symplectic, s is hamiltonian, maximal degree is 1. Use Darboux theorem to get

$$\sigma = dp^i \wedge dq_i + dc^\alpha \wedge d\mathcal{P}_\alpha \qquad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_\beta) = -1$$

The Hamiltonian for s (BRST charge)

$$\Omega = c^\alpha T_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta} \mathcal{P}_\gamma + \text{terms of degree } \geqslant 2 \text{ in } \mathcal{P}_\alpha$$

T_α – first-class constraints.

$$\{\Omega, \Omega\} = 0$$

These are defining relations of BFV formalism

Batalin, Fradkin, Vilkovisky, 1977

Quantization: Representation space $\mathcal{H}$: e.g. functions in q^i, c^α $(\hbar = 1)$

$$\widehat{p}_i = \frac{\partial}{\partial q^i}\,, \qquad \widehat{\mathcal{P}}_\alpha = \frac{\partial}{\partial c^\alpha}$$

$$\Omega \quad \rightarrow \quad \widehat{\Omega}\,, \qquad \widehat{\Omega}^2 = 0$$

Physical representation space: $H^0(\widehat{\Omega}, \mathcal{H})$
Physical observables: $H^0([\widehat{\Omega}, \cdot], \text{operator algebra})$
- so that one gets usual quantum mechanics in the cohomology (modulo subtleties)

Note: above is correct for time reparametrization invariant systems. Otherwise in addition one has Hamiltonian $\widehat{H}$ determining the evolution.

Gives us all the data of linear gauge PDE. If there is an inner product, it determines odd symplectic structure.

Example: relativistic particle

Pseudo-Riemanien manifold X with coordinates x^a, phase space: $T^*X\ \times$ ghost space $c, \mathcal{P}$. Standard Poisson bracket and BRST charge:

$$\{x^a, p_b\} = \delta^a_b\,, \quad \{c,b\} = 1\,, \qquad \Omega = cg^{ab}p_a p_b + m^2$$

Upon quantization:

$$\Phi = \phi(x) + c\chi(x)\,, \qquad \langle , \rangle = \int \sqrt{g}d^n x dc\,, \qquad \widehat{\Omega} = c(\nabla^2 + \ldots + m^2)$$

Equations of motion and Lagrangian:

$$(\nabla^2 + m^2)\phi(x) = 0\,, \qquad S_{BV} = S = \int \sqrt{g}d^n x\phi(\nabla^2 + m^2)\phi\,.$$

AKSZ sigma models

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994

M - supermanifold (target space) with coordinates Ψ^A:
Ghost degree – gh()
(odd)symplectic structure σ, gh$(\sigma) = n - 1$ and hence
(odd)Poisson bracket $\{\cdot\,,\,\cdot\}$, gh$(\{\cdot\,,\,\cdot\}) = -n + 1$
"BRST potential" $S_M(\Psi)$, gh$(S_M) = n$, master equation $\{S_M, S_M\} = 0$
(QP structure: $Q = \{\cdot\,, S_M\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)
Ghost degree gh( )
$\boldsymbol{d}$ – odd vector field, $\boldsymbol{d}^2 = 0$, gh$(\boldsymbol{d}) = 1$
Typically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \ldots n - 1$

$\Phi : \mathcal{X} \to M$. Fields: $\Psi^A(x,\theta) := \Phi^*(\Psi^A)$.

BV master action

$$S_{BV} = \int \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M) \right] \,, \qquad \mathrm{gh}(S_{BV}) = 0$$

χ is potential for $\sigma = d\chi$. In components:

$$S_{BV} = \int d^n x d^n \theta \left[\chi_A(\Psi(x,\theta)) \boldsymbol{d} \Psi^A(x,\theta) + S_M(\Psi(x,\theta)) \right]$$

BV antibracket

$$\Big(F, G\Big) = \int d^n x d^n \theta \left(\frac{\delta^R F}{\delta \Psi^A(x,\theta)} \sigma^{AB} \frac{\delta G}{\delta \Psi^B(x,\theta)} \right) , \qquad \mathrm{gh}\Big(,\Big) = 1$$

$\sigma^{AB}(\Psi)$ – components of the Poisson bivector.

Master equation:

$$\Big(S_{BV}, S_{BV}\Big) = 0 \,,$$

BRST differential:

$$s^{AKSZ}\Psi^A(x,\theta) = \boldsymbol{d}\Psi^A(x,\theta) - Q^A(\Psi(x,\theta))\,, \qquad Q^A = \left\{\Psi^A, S_M\right\}$$

Natural lift of Q and $\boldsymbol{d}$ to the space of maps.

Dynamical fields: those of vanishing ghost degree

$$\Psi^A(x,\theta) = \overset{0}{\Psi}{}^A(x) + \overset{1}{\Psi}{}^A_\mu(x)\theta^\mu + \dots \qquad \mathrm{gh}(\overset{k}{\Psi}{}^A_{\mu_1\dots\mu_k}) = \mathrm{gh}(\Psi^A) - k$$

If $\mathrm{gh}(\Psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\Psi}{}^A_{\mu_1\dots\mu_k}(x)$ is dynamical.

AKSZ equations of motion

$$\sigma_{AB}(\boldsymbol{d}\Psi^A - Q^A) = 0\,, \quad \Rightarrow \quad (\boldsymbol{d}\Psi^A(x,\theta) - Q^A(\Psi(x,\theta))) = 0$$

Recall: σ_{AB} is invertible. Interesting alternative: degenerate σ – presymplectic AKSZ *Alkalaev, M.G. 2013*

More invariantly, if $\Psi^A(x,\theta) = \Phi^*(\psi^A)$ the equations of motion read as:

$$\boldsymbol{d}\Phi^*(\psi^A) = \Phi^*(Q\psi^A) \quad \Leftrightarrow \quad \boldsymbol{d}\circ\Phi^* = \Phi^*\circ Q$$

so that Φ^* is a morphism of respective complexes. Pure gauge solutions are trivial morphisms, i.e. Φ^* of the form

$$\Phi^* = \boldsymbol{d}\circ\chi^* + \chi^*\circ Q$$

Note: strictly speaking one needs to extract equations and gauge symmetries for dynamical fields only

AKSZ at the level of equations of motion (nonlagrangian)

$$\{,\}\,,\, S_M \quad \Rightarrow \quad Q = Q^A \frac{\partial}{\partial \Psi^A} \quad Q^2 = 0\,.$$

I.e. target is a generic Q manifold.

target doesn't know dim X! (Recall gh$(S_M) = n = \dim X$)

If gh$(\Psi^A) \geqslant 0 \;\forall\; \Psi^A$ then BV-BRST extended FDA.

Otherwise BV-BRST extended FDA with constraints.

Examples:

Chern-Simons: *AKSZ, 1994*

Target space M:

$M = \mathfrak{g}[1]$, $\mathfrak{g}$ – Lie algebra with invariant inner product.

e_i –basis in $\mathfrak{g}$, C^i – coordinates on $\mathfrak{g}[1]$, $\mathrm{gh}(C^i) = 1$, $C = C^i e_i$

$$S_M = \frac{1}{6}\langle C, [C, C]\rangle\,, \qquad \left\{C^i, C^j\right\} = \langle e_i, e_j\rangle^{-1}$$

Source space:

$\mathcal{X} = T[1]X$, X – 3-dim manifold. Field content

$$C^i(x,\theta) = c^i(x) + \theta^\mu A^i_\mu(x) + \theta^\mu\theta^\nu A^{*i}_{\mu\nu} + (\theta)^3 c^{*i}$$

BV action

$$S_{BV} = \int (\frac{1}{2}\langle C, \boldsymbol{d}C\rangle + \frac{1}{6}\langle C, [C, C]\rangle) = \int \frac{1}{2}\langle A, \boldsymbol{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle) + \ldots$$

1d AKSZ systems: Target space M – Extended BFV phase space: $\{,\}$ – Poisson bracket, $S_M = \Omega$, Ω – BRST charge

Source space $\mathcal{X} = T[1](\mathbb{R}^1)$, coordinates t, θ

BV action *M.G., Damgaard, 1999*

$$S_{BV} = \int dtd\theta(\chi_A \boldsymbol{d}\psi^A + \Omega)$$

Integration over θ gives BV for the Hamiltoninan action

Fisch, Henneaux, 1989, Batalin, Fradkin 1988.

Example: coordinates on M: $\widetilde{c}, \widetilde{\mathcal{P}}, \widetilde{x}^\mu, \widetilde{p}_\mu$, BRST charge $\Omega = \widetilde{c}T(x,p)$,

$$S_{BV} = \int dtd\theta(\widetilde{p}_\mu \boldsymbol{d}\widetilde{x}^\mu + \widetilde{\mathcal{P}}\boldsymbol{d}\widetilde{c} + \widetilde{c}T) = \int dt(p_\mu \dot{x}^\mu + \lambda T) + \ldots$$

$$\widetilde{x}^\mu(t,\theta) = x^\mu(t) + \theta p^\mu_*(t)\,, \qquad \widetilde{p}_\mu(t,\theta) = p_\mu(t) + \theta x^*_\mu(t)\,,$$

$$\widetilde{c}(t,\theta) = c(t) + \theta\lambda(t)\,, \qquad \ldots$$

– Background-independent

– AKSZ is both Lagrangian and Hamiltonian

AKSZ model: $(M, S_M, \{,\})$ and $(\mathcal{X}, \boldsymbol{d})$.

Let $X = X_S \times \mathbb{R}^1$ *Barnich, M.G, 2003*

$$\Omega_{BFV} = \int_{X_S} [(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M)] \, , \qquad \mathrm{gh}(\Omega_{BFV}) = 1$$

$$\{\,\cdot\,,\,\cdot\,\}_{BFV} = \int d^{n-1}x d^{n-1}\theta \, \{\,\cdot\,,\,\cdot\,\} \qquad \{\Omega_{BFV}, \Omega_{BFV}\}_{BFV} = 0\,.$$

– Higher BRST charges *Cattaneo et. all. (2012)*

Similarly: $X_k \subset X$ – codimension-k submanifold

$$\Omega_{X_k} = \int_{X_k} (\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M))$$

In particular, $\Omega_{BFV} = \Omega_{X_S}$, $S_{BV} = \Omega_X$

– At the level of EOMs AKSZ is closely related to unfolded formalism of HS theories *Vasiliev 1988,. . .* and FDA approach of SUGRA *D'Auria, Fre,. . .*

– At the level of EOMs the same target space gives an AKSZ model for any $Z \subset X$ or even different X.
– (asymptotic) boundary values, e.g. in the context of AdS/CFT for HS theories *Vasiliev, 2012; Bekaert M.G. 2012*

– Locally in X and M: *Barnich, M.G. 2009*

$$H^g(s^{AKSZ}, \text{local functionals}) \cong H^{g+n}(Q, C^\infty(M))$$

The isomorphism sends $f \in C^\infty(M)$ to functional $F = \int \Phi^*(f)$.

Generalization to nontrivial X, *G.Bonavolonta, A.Kotov, 2013*

– For M finite dimensional and $n > 1$ – the model is topological. What about non-topological? Examples of non-topological systems whose equations of motion have the form of FDA in the context of HS theories

Vasiliev 1988,. . . .

AKSZ form of gauge PDE

Gauge PDE $(\mathcal{E}, s, d_{\mathsf{h}})$.

Rename $dx^a \to \xi^a$ and rename $x^a \to z^a$. Setting $\mathrm{gh}(\xi^a) = 1$ consider $\mathcal{E}$ as a Q-manifold $(\mathcal{E}, Q)$ with

$$Q = -d_{\mathsf{h}} + s$$

The total differential familiar in the local BRST cohomology

Stora 1983, Barnich, Brandt, Henneaux 1993,. . . .

Take $\mathcal{X} = T[1]X$ with coordinates x^μ, θ^μ and consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$.

Note that now z^a is promoted to a dynamical field $z^a(x)$ and ξ^a to dynamical field $e^a_\mu(x) dx^\mu$ and ghost field $\overset{0}{\xi}{}^a$

In fact: we are dealing with parametrized version.
$z^a(x)$ – space-time coordinates understood as fields
$e^a_\mu(x)$ – frame field components.

Gauge transf. for z^a: $\delta z^a = \xi^a + \ldots$. i.e. d_{h} is the BRST differential implementing reparametrization invariance.
Gauge condition $z^a = \delta^a_\mu x^\mu$ gives un-parametrized version where $e^a_\mu = \delta^a_\mu$. In the simplest case $s = 0$ the EOM's take the form

$$\frac{\partial}{\partial x^a} \Psi^A(x) - (\partial^T_a \Psi^A)(x, \theta) = 0$$

where Ψ^A are all the coordinates on $\mathcal{E}$. This is an equivalent representation of the original system.

Parent formulation: AKSZ sigma model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$, where $Q = -d_{\mathsf{h}} + s$.

(Locally on $\mathcal{X}$, $\mathcal{E}$) parent formulation is equivalent to the parameterized form of $(\mathcal{E}, s, d_{\mathsf{h}})$. *G. Barnich, M.G. 2010*

In the case of linear system the parent construction amounts to *Fedosov*-type extension applied to the BRST first-quantized complex of the theory. *Barnich, M.G., Semikhatov, Tipunin, 2004*.

If the theory is diffeomorphism-invariant $(\mathcal{E}, -d_{\mathsf{h}} + s) \cong (\mathcal{E}_0, s) \times (T[1]X, d)$ and the parent formulation is equivalent to AKSZ with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}_0, s)$) (space-time is "gauged away")

Equivalence: homotopy equivalent target space Q-manifolds lead to equivalent gauge field theories.

Example: contractible pairs for Q: suppose by local invertible change of coordinates:

$$Qw^a = v^a, \qquad Q\psi^\alpha = Q^\alpha(\psi)$$

then w^a, v^a are contractible pairs. Their elimination results in the reduced Q-manifold $(\widetilde{Q}, \widetilde{\mathcal{E}})$.
Often one can arrive at "minimal" Q-manifold associated the gauge system
Known as manifold of generalized connections and tensor fields.
Brandt, 1996
Similar formulations are known in the HS context for a long time
Vasiliev 1988,. . .

What contractible pairs for Q look like in field theoretical terms? For the AKSZ model trivial pairs give rise to generalized auxiliary fields: These comprise usual auxiliary fields, algebraically pure gauge (Stueckelberg) fields, their associated ghosts/antifields analogous fields in the sector of reducibility relations.

Lagrangian: *Dresse, Grégoire, Henneaux, 1990*
EOM: *Barnich, M.G., Semikhatov, Tipunin, 2004*

Nonlocal "equivalence": if $X = X_0 \times \mathbb{R}^k$ then AKSZ model on X is "closely related" to that on X_0. Can be "pulled back". Boundary values.

CFT with HS symmetry

Consider as a simplest and standard example free conformal scalar

$$\Box \phi = 0$$

Symmetries:

$$[\Box, A] = B\Box \,, \qquad A, B- \text{ differential operators}$$

Associated conserved HS currents:

$$J_{a_1 \ldots a_s} = \bar{\phi} \partial_{a_1} \ldots \partial_{a_s} \phi + \ldots$$

Sources:

$$\langle h, J \rangle = \sum_s J_{a_1 \ldots a_s} h^{a_1 \ldots a_s}$$

$$\partial_{a_1} J^{a_1 a_2 \ldots a_s} = 0 \quad \rightarrow \quad \delta h^{a_1 \ldots a_s} = \partial^{(a_1} \xi^{a_2 \ldots a_s)} - \text{traces}$$

These sources are infinitesimal in the sense that the action

$$S[\phi, h] = \langle \phi, \Box\phi \rangle + \langle J, h \rangle$$

is only invariant under gauge transformations

$$\delta h = \partial \cdot \xi + \eta\omega \,, \qquad \delta\phi = 0 \,.$$

at $\Box\phi = 0$. This symmetry is not enough to fix the correlation functions.

The enhanced gauge symmetry can be found using the different base for the currents and the sources (background fields) *Segal 2002*

$$S[\phi, F] = \langle \phi, F\phi \rangle \,,$$

$$F = \sum_s F^{a_1 \ldots a_s} \partial_{a_1} \ldots \partial_{a_s} \,, \qquad \delta F = FU + U^\dagger F \,, \quad U = U(x, \partial) \,.$$

Writing $F = \Box + h'$ note that h' is related to h through a nontrivial and nonlinear redefinition.

Integrating out the scalar results in the effective action:

$$e^{W[F]} = \int D\phi\, D\bar{\phi}\, e^{-\int \langle \phi, F\phi \rangle}\,, \qquad W[F] = -\mathrm{tr}\log F\,,$$

Its invariance with respect to

$$\delta F = FU + U^{\dagger}F\,, \quad U = U(x, \partial)\,.$$

encodes HS invariance of the correlation functions of $J_{a\ldots}$, which in known to fix them *Maldacena, Zhiboedov*. HS algebra show up as the algebra of Killings.

It follows, all the information about free CFT is encoded in the gauge theory of background fields (finite sources).

- These fields are off-shell
- Subject to nonlinear gauge symmetries

Usual understanding of HS holography:

Nonlinear HS in the bulk $\Leftrightarrow$ CFT with HS symmetry

The idea is to replace it with:

Nonlinear HS in the bulk $\Leftrightarrow$ Nonlinear background fields

It's easier to relate objects of the same nature.

Somewhat implicitly, was employed in the case of theory of symmetric HS fields (Vasiliev theory) in *Bekaert, M.G., Skvortsov, 2017*
Earlier relevant developments: *Alkalaev, M.G. Skvortsov 2014, Bekaret, MG 2013, MG 2012, MG 2006*

Background fields from constrained system

Given a quantum constrained system:

$$\widehat{F}_i|\Phi\rangle = 0\,, \qquad \text{(More generally: } F_a|\Phi\rangle = 0\,, \quad |\Phi\rangle \sim F_\alpha|\Xi\rangle)$$

The consistency condition (switching to star product notations):

$$[F_i, F_j]_\star = U_{ij}^k \star F_k$$

Natural equivalence transformations for the constraints:

$$\delta F_i = \lambda_i^j \star F_j + [\epsilon, F_i]_\star$$

$F_i(x,p)$ can be seen as a generating function of background fields:

$$F_i = F_i(x) + F_i^a(x)p_a + F_i^{ab}(x)p_a p_b + \ldots$$

while the above consistency and the equivalence, as resp. equations of motion and gauge symmetries.

Background fields for a scalar

Phase space:

$$[x^a, p_b]_\star = \delta^a_b$$

First class constraint $F(x,p)$. Although the consistency condition is trivial the gauge symmetries are

$$\delta F = [F, \xi]_\star + F \star \omega$$

These are precisely gauge symmetries for background fields for a conformal scalar *Segal 2002*.

Upon linearizing around $F_0 = p^2$ one gets

$$\delta F = p \cdot \frac{\partial}{\partial x} \xi + p^2 \star \omega$$

Off-shell conformal fields *Fradkin, Tseytlin 1985*. HS algebra arise as that of global reducibility parameters.

BV-like description

M.G. 2006

Associate a pair of ghosts c^i, b_j, $[c^i, b_j] = \delta^i_j$ to each constraint F_i. Denote by $\mathcal{A}$ the $\star$-product algebra generated by $x^a, p_b, c^\alpha, b_\alpha$. A generic element of $\mathcal{A}$

$$\Phi = \Phi^A(x) e_A\,, \qquad \Phi = \phi(x) + \phi^a p_a + c^\alpha \phi_\alpha + \ldots$$

Equations of motion and gauge symmeries:

$$[\Omega, \Omega]_\star = 0\,, \qquad \delta\Omega = [\Omega, \Xi]_\star\,, \quad \mathrm{gh}(\Omega) = 1\,, \quad \mathrm{gh}(\Xi) = 0$$

Q-manifold picture: promote Φ^A to fields Ψ^A with $\mathrm{gh}(\Psi^A) = 1 - \mathrm{gh}(e_A)$. String field:

$$\Psi = \Psi^A(x) e_A\,, \qquad \mathrm{gh}(\Psi) = 1$$

BRST differential:

$$s\Psi = \frac{1}{2}[\Psi, \Psi]_\star\,,$$

Ambient description of the conformal scalar

In terms of ambient space $\mathbb{R}^{d+2}$ with coordinates X^A, flat metric η^{AB}:

$$(X \cdot \partial_X + \frac{d-2}{2})\Psi = 0\,, \qquad\qquad \partial_X \cdot \partial_X \Psi = 0\,,$$

$$\Psi \sim \Psi + X^2\alpha\,,$$

The above operators

$$X^2\,, \qquad \partial_X^2\,, \qquad (X \cdot \partial_X + \tfrac{d+2}{2})\,,$$

form $sp(2)$.

Background Fields in Ambient Space

Phase space: $\qquad [X^A, P_B]_\star = \delta^A_B \,,$
Generating functions:

$$F_i(X,P)\,, \qquad i = \{+, 0, -\}$$

Equations:

$$[F_i, F_j]_\star = U_{ij}^k \star F_k\,, \qquad U_{ij}^k - sp(2) \text{ structure constants}$$

Gauge symmetries: $\delta F_i = \lambda_i^j \star F_j + [\epsilon, F_i]_\star$
HS extension of the *Fefferman-Graham (1985)* construction. Natural vacuum solution:

$$F^0_+ = \frac{1}{2}P^2\,, \quad F^0_0 = X \cdot P\,, \quad F^0_- = \frac{1}{2}X^2\,,$$

Linearization around just reproduces the background conformal fields.

Bulk fileds: free Level

Field content:

$$\phi = \sum_s \phi^{A_1 \dots A_s}(x) P_{A_1} \dots P_{A_s}$$

Ambient space description:

$$(X \cdot \partial_P)\phi = 0\,, \quad (X \cdot \partial_X - P \cdot \partial_P + 2)\phi = 0$$
$$(\partial_X \cdot \partial_X)\phi = (\partial_X \cdot \partial_P)\phi = (\partial_P \cdot \partial_P)\phi = 0$$
$$\delta\phi = (P \cdot \partial_X)\xi$$

Boundary values of these fields coincide with the linearized background conformal fields (modulo holographic anomaly in even d). Obvious using the technique of *Bekaert, MG 2012*.

Holographic reconstruction

The theory is determined by its on-shell gauge transformations

In the case at hand background fields on the boundary are 1:1 with on-shell bulk fields. Moreover, we know nonlinear gauge transformations for background fields!

We can in some sense reconstruct the bulk theory.

In the ambient approach bulk/boundary relation amounts to considering the same system either around $X^2 = -1$ or $X^2 = 0$ *Bekaert, MG 2012*.

A proper language is that bulk theory and boundary theory "live" around two different vacua (in some precise sense X is a background field)

The proposal for HS theory is the same $sp(2)$ system

$$[F_i, F_j]_\star = U_{ij}^k \star F_k\,, \qquad \delta F_i = \lambda_i^j \star F_j + [\epsilon, F_i]_\star$$

considered in the vicinity of the hyperboloid $X^2 = -1$.

Clearly requires regularization if considered around the vacuum $F_-^0 = X^2$. Can be analyzed by switching to the parent formulation.

Parent formulation

Field content:

$$A = dx^\mu A_\mu(x|Y,P)\,, \qquad F_i = F_i(x|Y,P)\,,$$

"Internal" ambient space:

$$[Y^A, P_B]_\star = \delta^A_B\,,$$

Full on-shell system:

$$\begin{aligned}
dA - \frac{1}{2}[A,A]_\star &= u^i \star F_i\,, & \delta A &= d\xi - [A,\xi]_\star + \lambda^j \star F_j\,,\\
dF_i - [A,F_i]_\star &= u_i^j \star F_j\,, & \delta F_i &= [\xi, F_i]_\star + \lambda_i^j \star F_j\,,\\
[F_i,F_j]_\star - C^k_{ij} F_k &= u^k_{ij} \star F_k\,.
\end{aligned}$$

Vacuum solution:

$$A^0 = dx^\mu \omega_{\mu A}{}^B T_B^A\,, \qquad V^A = const^A \text{– compensator}$$

$$F_-^0 = \frac{1}{2}(Y+V)\cdot(Y+V)\,, \quad F_+^0 = \frac{1}{2}P\cdot P\,, \quad F_0^0 = (Y+V)\cdot P\,,$$

$$T^{AB} = -(Y^A + V^A)\cdot P^B - (A \rightleftarrows B)\,,$$

Cartan description of conformally-flat geometry: $\omega_A^B -$ flat $o(d,2)$-connection, V^A, $V^2 = 0$

For AdS-geometry: $V^2 = -1$ and all fields are defined on AdS

Passage to the bulk *Bekaert, M.G. 2012*

$$A, F_i(x^a, Y, P) \quad \rightarrow \quad A, F_i(x^\mu, Y, P)$$

$$\nabla_{Conf} \rightarrow \nabla_{AdS}$$

$$V^A_{Conf} \rightarrow V^A_{AdS} \qquad (V^2_{AdS} = -1, \quad V^2_{Conf} = 0)$$

Disgregarding gauge symmetry:

$$\delta f_i = \lambda_i^j \star F_j^0 \,, \qquad \delta a = \lambda^i F_i^0$$

the linearized system (after homological reduction):

$$\begin{aligned} D_0 a &= 0\,, & \delta a &= D_0 \xi\,,\\ D_0 f_+ &= [a, F_+^0]_\star\,, & \delta f_+ &= [F_+^0, \xi]_\star\,,\\ [F_-^0, f_+]_\star &= [F_0^0, f_+]_\star - 2 f_+ = 0\,, \quad & [F_-^0, a]_\star &= [F_0^0, a]_\star = 0\,, \end{aligned}$$

If a, f_+ were totally traceless this is precisely the system from *Barnich, MG, 2006* describing the Fronsdal fields on AdS.

If we could use gauge symmetry:

$$\delta f_i = \lambda_i^j \star F_j^0\,, \qquad \delta a = \lambda^i \star F_i^0$$

to set f_i, a totally traceless, we would conclude that the proposed theory properly describes free limit.

For the linearized system it's true provided we pick a proper functional class $\mathfrak{C}$: polynomials in P, formal series in Y such that

$$(\partial_Y \cdot \partial_Y)^l \phi = 0$$

Then there is a twisted traceless projector Π':

$$\phi = \phi_0 + (Y+V_0)^2\phi_{10} + (Y+V)\cdot P\phi_{11} + \dots\,, \qquad \phi_{\dots} - \text{totally traceless}$$

$$\Pi'\phi = \phi_0$$

In this class $Tr.(a) = Tr.(f_i) = 0$ is a legitimate gauge condition.

HS-flat connection

The system admits vacuum solution (belonging to $\mathfrak{C}$):

$$F_i^0, \qquad A^0(x, P, Y, \theta)$$

$$[F_i^0, A^0]_\star = 0 \qquad dA^0 - \frac{1}{2}[A^0, A^0]_\star = u^i \star F_i^0\,,$$

i.e. A^0 is a flat connection of Type-B HS algebra.

Higher spin algebra hs:

$$\mathsf{hs} = \{\mathsf{a} \in \mathfrak{C} : [\mathsf{a}, F_i^0]_\star = 0\,, \quad \mathsf{a} \sim \mathsf{a} + \lambda^i \star F_i^0\}\,,$$

Representatives can be taken traceless. $\star$-descends to hs.

Linearized system (after homological reduction):

$$
\begin{aligned}
D_0 a &= 0\,, & D_0 f_+ &= [a, F^0_+]_\star\,,\\
\delta a &= D_0 \xi\,, & \delta f_+ &= [\xi, F^0_+]_\star\,,\\
[F^0_-, f_+]_\star = [F^0_-, a]_\star &= 0\,, & [F^0_0, f_+]_\star - 2 f_+ &= [F^0_0, a]_\star = 0\,,
\end{aligned}
$$

where $D_0 \bullet \equiv d\bullet - \Pi' [A^0, \bullet]_\star$.

This gives a concise formulation of the multiplet of fields propagating on the background of generic flat connection of the Type-A higher spin algebra.

This can be reduced to unfolded form (e.g. using the homological reduction of *Barnich, MG (2006)*). It should have the structure:

$$d\bar{a} + \Pi'([A_0, \bar{a}]) = \mu(A_0, A_0, C) \qquad \text{note: } A_0 \in \mathsf{hs}_B$$

$P \cdot \partial_Y \bar{a} = 0$ and C parametrises the quotient $f_+ \sim f_+ + P \cdot \partial_Y \epsilon$.

Recent work by *Sharapov, Skvortsov* shows that such μ is a Hochshild cocycle of the HS algebra and it fully determines the complete deformation (the deformation is unobstructed due to the absence of higher cohomology).

This gives an extra argument in support that the proposed system "knows everything" about the HS gravity.

Conclusions

- Parent formulation – AKSZ formulation of generic gauge PDE. In particular, systematic way to "unfold" any gauge theory.
- Concise consistent system of equations describing formal Type-A (totally symmetric) HS theory, reproducing all the structures of Vasiliev theory.
- Straightforward generalization to extended HS theory containing partially-massless fields *Bekaert, M.G. 2013*
- Straightforward generalization to a new HS theory (Type-B) dual to conformal spinor on the boundary *M.G. Skvortsov 2018*

- $sp(2) \to osp(1|2)$
- spectrum (hook-type fields):

$$\phi_{a_1 \dots a_s, b_1 \dots b_q}(x) \sim \begin{array}{|c|c|}\hline & \qquad s \qquad \\ \hline q & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

- byproduct: nonlinear theory of conformal HS fields. Being supplemented with divergent part of the effective action for spinor should produce CHS nonlinear HS theory (Type-B version of CHS theory of *Segal, Tseytlin, 2002*).

- Build in terms of boundary conformal spinor. HS holography automatically built in thanks to the ambient formalism Nonlinear CHS fields are reproduced on the boundary. Classical version of holographic reconstruction?

- HS (and $osp(1|2)$) generalization of the *Fefferman-Graham* construction. Proper language for HS geometry?

- Unifies metric-like and frame-like formalism. In particular, F_+ is an ambient version of the metric-like HS field

- Likely to provide a framework for studying nonlocality issue at more invariant level