EIT: anisotropy within reach via curved interfaces

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Outline

Electrical Impedance Tomography (Calderón's problem).

- Uniqueness in EIT
- The anisotropic case: obstruction to uniqueness.

Global uniqueness for anisotropic piecewise constant conductivities.

- The local Neumann-to-Dirichlet map and the Neumann kernel.
- Unique determination of the conductivity/metric.
- An example of non uniqueness.

- $\Omega \subset \mathbb{R}^n$ domain ($n \geq 2$);
- Conductivity Equation:

$$\operatorname{div}(\sigma \nabla u) = 0 \qquad \text{in} \quad \Omega, \tag{1}$$

$\sigma = \sigma(x)$, $x \in \Omega$, is a symmetric, positive definite matrix and it represents the electric conductivity of Ω.

- Dirichlet-to-Neumann map:

$$\Lambda_\sigma : u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \sigma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega),$$

for any $u \in H^1(\Omega)$ solution to (1).

Problem:

Can we determine σ from Λ_σ?

We want to study properties of the map

$$\Lambda : \sigma \longrightarrow \Lambda_\sigma$$

- injectivity of Λ (uniqueness) $\longleftarrow$
- Continuity of Λ and its inverse if it exists (stability)
- What is the range of Λ? (characterization problem)
- Formula to recover σ from Λ_σ (reconstruction)
- Give an approximate numerical algorithm to find an approximation of the conductivity given a finite number of voltage and current measurements at the boundary (numerical reconstruction)

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Calderón's problem:

the current formulation of the problem is due to Calderón in 1980 and his seminal paper opened the way to the solution to the uniqueness issue.

Isotropic case, $n \geq 3$

- σ piecewise real analytic (Kohn - Vogelius ,1984)
- $\sigma \in C^2(\overline{\Omega})$ (Sylvester - Uhlmann, 1987)
- $\sigma \in C^{1+\varepsilon}(\overline{\Omega})$, σ conormal (Greenleaf - Lassas - Uhlmann, 2003)
- $\sigma \in C^1(\overline{\Omega})$ (Haberman - Tataru, 2013)
- $\sigma \in W^{1,n}(\Omega), n = 3, 4, 5$ (Haberman, 2015)
- σ Lipschitz, any dimension $n \geq 3$ (Caro - Rogers, 2015).

Isotropic case, $n = 2$

- $\sigma \in C^2(\overline{\Omega})$ (Nachman, 1996)
- σ Lipschitz (Brown-Uhlmann, 1997)
- $\sigma \in L^\infty(\Omega)$ (Aslala-Päivärinta, 2006).

Remark for $n \geq 3$:

- Independent study of the problem from Calderón's formulation in the geophysical setting (Druskin, 1983, 1985, 1998)
- Lipschitz stability for σ piecewise constant (Alessandrini-Vessella, 2005).

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Anisotropic case: σ matrix

- Tartar's observation: If ψ is a diffeomorphism

$$\psi : \bar{\Omega} \to \bar{\Omega}$$

with

$$\psi(x) = x, \qquad \forall x \in \partial\Omega,$$

then

$$\sigma; \qquad \tilde{\sigma} = \frac{(D\psi)\sigma(D\psi)^T}{\det D\psi} \circ \psi^{-1}$$

have the same Dirichlet-to-Neumann map i.e. $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$.

Uniqueness up to diffeomorphism

Theorem ($n \geq 3$) (Lassas-Uhlmann 2001, Lassas-Taylor-Uhlmann 2003)

$(M, g_i), i = 1, 2$, real analytic, connected, compact, Riemannian manifold with boundary. Let $\Sigma \subset M$, Σ open. Assume

$$\Lambda^{\Sigma}_{g_1} = \Lambda^{\Sigma}_{g_2},$$

then $\exists\ \psi$ diffeomorphism, $\psi\Big|_{\Sigma} = I$ such that

$$g_1 = \psi^* g_2.$$

Lee-Uhlmann 1989; Belishev 2003; Guillarmou-Sa Barreto 2007 for Einstein manifolds.

Theorem ($n = 2$) (Lassas-Uhlmann 2001)

$(M, g_i), i = 1, 2$, connected Riemannian manifold with boundary. Let $\Sigma \subset M$, Σ open. Assume

$$\Lambda_{g_1}^{\Sigma} = \Lambda_{g_1}^{\Sigma},$$

then $\exists\ \psi$ diffeomorphism, $\psi\Big|_{\Sigma} = I$ and $\alpha > 0$, $\alpha\Big|_{\Sigma} = 1$ such that

$$g_1 = \alpha\psi^* g_2.$$

$\Omega \subseteq \mathbb{R}^2$

- $\sigma \in C^2(\overline{\Omega})$ (Nachman,1996)
- σ Lipschitz (Sun-Uhlmann, 2003)
- $\sigma \in L^\infty(\Omega)$ (Astala-Lassas-Päivärinta, 2006)
- $\sigma \in C^\infty(\overline{\Omega})$ with partial data (Imanuvilov-Uhlmann-Yamamoto, 2011).

Uniqueness with *a-priori* information about σ

Assume that σ is *a-priori* known to depend on a restricted number of unknown spatially dependent parameters.

- Kohn-Vogelius (1984): all the eigenvalues of σ are known except for one eigenvalue
- Alessandrini (1990): stability at the boundary for $\sigma(x) = \sigma(a(x))$
- Lionheart (1997): uniqueness at the boundary for $\sigma(x) = a(x)\sigma_0(x)$
- Alessandrini-G. (2001, 2009) : stability at the boundary $\sigma(x) = \sigma(x, a(x))$ for global and partial data
- Alessandrini-Cabib (2007): uniqueness for divergence free σ (n=2)
- G.-Lionheart (2009): stability at the boundary for $g(x) = g(x, a(x))$ Riemannian metric
- G.-Sincich (2015): Lipschitz stability for $\sigma(x) = a(x)\sigma_0(x)$, a piecewise constant, σ_0 Lipschitz

Uniqueness for the potential q in the Schrödinger equation

(M, g) compact Riemannian manifold with boundary ∂M

$$\left(-\Delta_g + q\right) u = 0 \quad \text{in } M$$

Assume that g is *a-priori* known to be of type

$$(M, g) \subset\subset (\mathbb{R} \times M_0, g), \text{ where } g = c(e \oplus g_0), \quad c > 0$$

then Λ_q uniquelly determines q.

- Dos Santos-Kenig-Salo-Uhlmann, 2009
- Dos Santos-Kurylev-Lassas-Salo, 2016.

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The Neumann-to-Dirichlet map

$\Omega \subseteq \mathbb{R}^n$ domain, $n \geq 3$;

- $$ {}_0H^{\frac{1}{2}}(\partial\Omega) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) \middle| \int_{\partial\Omega} f = 0 \right\}, $$
$$ {}_0H^{-\frac{1}{2}}(\partial\Omega) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) \middle| \langle \psi, 1 \rangle = 0 \right\}. $$
- The Neumann-to-Dirichlet (N-D) map associated to σ,
$$ \mathcal{N}_\sigma : {}_0H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow {}_0H^{\frac{1}{2}}(\partial\Omega) $$
is defined by
$$ \mathcal{N}_\sigma = \left(\Lambda_\sigma \big|_{{}_0H^{\frac{1}{2}}(\partial\Omega)} \right)^{-1}. $$

- $\mathcal{N}_\sigma$ is the selfadjoint operator that satisfies:

$$\langle \psi, \mathcal{N}_\sigma \psi \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla u(x)\, dx,$$

$\forall \psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to

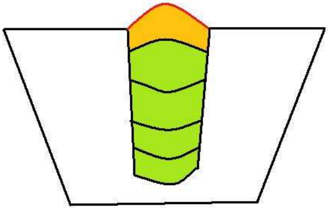
$$\begin{cases} \mathrm{div}(\sigma \nabla u) = 0, & \text{in} \quad \Omega, \\ \sigma \nabla u \cdot \nu|_{\partial\Omega} = \psi, & \text{on} \quad \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u = 0. \end{cases}$$

- $\mathcal{N}_\sigma \quad \longrightarrow \quad \mathcal{N}_\sigma^\Sigma, \qquad \Sigma$ open portion of $\partial\Omega$ (local data).

$$\mathcal{N}_\sigma^\Sigma(\psi), \quad \forall \psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega), \quad \mathrm{supp}(\psi) \subset\subset \Sigma.$$

Theorem (Alessandrini-de Hoop-G. (2016)):

- $\Omega \subset \mathbb{R}^n$ domain, $n \geq 3$
- $\partial\Omega$ of Lipschitz class
- $\Sigma \subset \partial\Omega$ $C^{1,\alpha}$ non-flat open portion (for the measurements)
- Ω is partitioned into N subdomains, $D_1, \ldots, D_N$ with Lipschitz boundaries
- The geometry of D_j, $j = 1, \ldots, N$ is assumed known
- Σ_k interface between D_{k-1} and D_k is $C^{1,\alpha}$ non-flat, $\forall k$



- $\sigma^{(i)}$, $i = 1, 2$ conductivity of type:

$$\sigma^{(i)}(x) = \sum_{j=1}^{N} \sigma_j^{(i)} \chi_{D_j}(x) \qquad x \in \Omega,\ i = 1, 2. \tag{2}$$

where $\sigma_j^{(i)} \in Sym_n$ satisfying a condition of uniform ellipticity.
Then:

$$\mathcal{N}^{\Sigma}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma}_{\sigma^{(2)}} \quad \Rightarrow \quad \sigma^{(1)} = \sigma^{(2)}, \qquad \text{in} \quad \Omega.$$

Lee-Uhlmann, 1989

- $\Omega \subset \mathbb{R}^n$ domain, $n \geq 3$
- $\sigma(x) = \{\sigma_{ij}(x)\}_{i,j=1,\dots,n}$, $x \in \Omega$ symmetric, positive definite
- $$g = (\det \sigma)^{\frac{1}{n-2}} \sigma^{-1}$$
- Consider the Riemannian manifold (Ω, g), then:

$$\frac{1}{\sqrt{detg}} L = \Delta_g \quad \text{operatore di Laplace-Beltrami}$$

- In dimension $n > 2$ we have:

$$\sigma \quad \longleftrightarrow \quad g$$

The Neumann kernel for Ω

$\forall y \in \Omega$,

$$\begin{cases} \operatorname{div}(\sigma(\cdot)\nabla N_\sigma(\cdot, y)) = -\delta(\cdot - y), & \text{in} \quad \Omega \\ \sigma\nabla N_\sigma(\cdot, y) \cdot \nu = -\frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega. \end{cases}$$

Lemma:

Let $y_0 \in \partial\Omega$ and suppose that $\partial\Omega$ is of class class $C^{1,\alpha}$ in a neighborhood of y_0. If σ is C^α in a neighborhood of y_0 in $\overline{\Omega}$, then:

$$N_\sigma(x,y_0), \quad \partial\Omega \ni x \sim y_0 \longrightarrow g_{(n-1)}(y_0) = \{g(y_0)v_i \cdot v_j\}_{i,j=1,\dots,(n-1)},$$

where $\{v_1, \dots, v_{n-1}\}$ is a basis of $T_{y_0}(\partial\Omega)$.

Theorem:

Let $y_0 \in \partial\Omega$ e suppose there is a neighborhood $\mathcal{U}$ of y_0 such that $\partial\Omega \cap \mathcal{U}$ is a portion of class $C^{1,\alpha}$, $0 < \alpha < 1$, of $\partial\Omega$ and $\sigma \in C^{\alpha}(\mathcal{U} \cap \overline{\Omega})$. Then the Neumann's kernel $N_\sigma(\cdot, y_0)$ satisfies:

$$\begin{aligned} N_\sigma(x,y_0) &= 2C_n \left(g(y_0)(x - y_0)\cdot(x - y_0)\right)^{\frac{2-n}{2}} \\ &+ O(|x - y_0|^{2-n+\alpha}), \end{aligned}$$

for $x \to y_0$, $x \in \overline{\Omega} \setminus \{y_0\}$.

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Unique determination of g on D_1

Lemma:

Let Σ be an open portion of $\partial\Omega$ of class $C^{1,\alpha}$ and non flat . If σ is constant near $y_0 \in \Sigma$, then:

$$N_\sigma(x,y) \quad \forall x,y \in \Sigma \text{ near } y_0 \longrightarrow g(y_0) \longrightarrow \sigma(y_0).$$

Proof (sketch):

- $\{e_1, \dots, e_n\}$ canonical basis of $\mathbb{R}^n$
- $T_{y_0}(\partial\Omega) = \Pi_n = < e_1, \dots, e_{n-1} >$
- $\nu(y_0) = -e_n$.

Σ is non flat near $y_0 \Rightarrow \exists y \in \Sigma$ near y_0 where $\nu(y)$ slightly deflects from $\nu(y_0) = -e_n \Rightarrow \exists y \in \Sigma$, $\exists \varepsilon \neq 0$ s.t:

$$\nu(y) = \frac{1}{\sqrt{1+\varepsilon^2}} \left(-e_n + \varepsilon e_{n-1}\right).$$

Alternative:

(a) The deflection of $\nu(y)$ with respect to $\nu(y_0)$ is in the e_{n-1} direction, $\forall y \sim y_0$.

(b) $\exists \widetilde{y} \in \Sigma$, $\widetilde{y} \sim y_0$ s.t. the deflection of $\nu(\widetilde{y})$ with respect to $\nu(y_0) = -e_n$ is in a direction indipendent from $e_{n-1} \Rightarrow$ $\exists \alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ t.c:

$$\nu(\widetilde{y}) = \frac{1}{\sqrt{1+\alpha^2+\beta^2}} \left(-e_n + \alpha e_{n-2} + \beta e_{n-1}\right).$$

GOAL: to show that in both cases (a) and (b), $g(y_0)$ is uniquely determined.
Denote

$$g = g(y_0)$$

(a). Orthonormal basis for $T_{y_0}(\Sigma)$:

$$\left\{ e_1, \dots e_{n-2}, \frac{1}{\sqrt{1+\varepsilon^2}}\, e_{n-1} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\, e_n \right\}.$$

Let $y = y(t)$, $0 < t < \varepsilon$ be a continuous path along Σ s.t. $y(0) = y_0$, $y(\varepsilon) = y$, $g(y(t)) = g$, $0 < t < \varepsilon$ and s.t. an orthonormal basis for $T_{y(t)}(\Sigma)$ is:

$$\left\{ e_1, \dots e_{n-2}, \frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n \right\}.$$

$\forall t$, $0 < t < \varepsilon$, the functions:

$$ge_i \cdot \left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right),$$
$$g\left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right) \cdot \left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right)$$

are known for any $i = 1, \ldots, n-2$, therefore:

- $$g_{i,\,n-1} + t g_{i,\,n}$$
  is known $\forall t$, $0 < t < \varepsilon$, for any $i = 1, \ldots, n-2$
- $$g_{n-1,\,n-1} + 2t g_{n-1,\,n} + t^2 g_{n,n}$$
  is known $\forall t$, $0 < t < \varepsilon$. □

$\mathcal{N}_\sigma^\Sigma \longrightarrow$ asymptotic behaviour of di $N_\sigma(x, y)$, $x, y \in \Sigma$.

- $$K_\sigma(x, y, w, z) = N_\sigma(x, y) - N_\sigma(x, w) - N_\sigma(z, y) + N_\sigma(z, w),$$
  $\forall x, y, w, z \in \Sigma$.

- If we fix $w, z \in \Sigma$, then $K_\sigma(x, y, z, w)$ and $N_\sigma(x, y)$ per $x \to y$ have the same asymptotic behaviour.

Lemma:

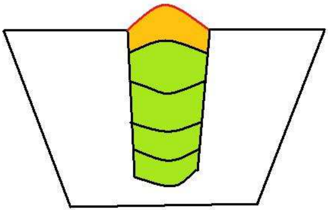
$$\mathcal{N}_\sigma^\Sigma \longrightarrow K_\sigma(x, y, z, w), \qquad \forall x, y, w, z \in \Sigma.$$

Proof of the main result:

- $\sigma^{(i)} \in Sym_n$, piecewise contant matrix on Ω satisfying a uniform ellipticity condition, $i = 1, 2$
- $$\mathcal{N}^{\Sigma_1}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma_1}_{\sigma^{(2)}}$$

  $\Rightarrow$

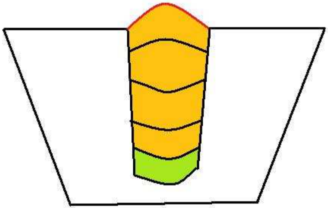
$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{on} \quad D_1.$$



- Induction: Let D_K be a subdomain of Ω, $K \neq 1$ and consider the chain of subdomains

$$D_1, \dots, D_K$$

connecting D_1 with D_K through domains with non flat $C^{1,\alpha}$ interfaces.



Assume that:

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{su} \quad D_i, \qquad \forall i, \quad 1 \leq i \leq K-1$$

and show that:

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{su} \quad D_K.$$

$$D = \left(\bigcup_{i=1}^{K-1} \overline{D_i} \right)^{\circ}; \qquad E = \Omega \setminus \overline{D}.$$

- Claim:

$$\left.\begin{array}{ll} \mathcal{N}^{\Sigma_1}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma_1}_{\sigma^{(2)}} & \\ \sigma^{(1)} = \sigma^{(2)} & \text{on} \quad D \end{array}\right\} \Rightarrow \mathcal{N}^{\Sigma_K}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma_K}_{\sigma^{(2)}}$$

  - $\mathcal{N}^{\Sigma_K}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma_K}_{\sigma^{(2)}} \Rightarrow$

$$\sigma^{(1)}(x) = \sigma^{(2)}(x), \qquad \text{for any} \quad x \in \Sigma_K$$

$\Rightarrow$

  -

$$\sigma^{(1)}(x) = \sigma^{(2)}(x), \qquad \text{for any} \quad x \in D_K.$$

Claim:

$$\left.\begin{array}{ll} \mathcal{N}_{\sigma^{(1)}}^{\Sigma_1} = \mathcal{N}_{\sigma^{(2)}}^{\Sigma_1} & \\ \sigma^{(1)} = \sigma^{(2)} & \text{on} \quad D \end{array}\right\} \Rightarrow \mathcal{N}_{\sigma^{(1)}}^{\Sigma_K} = \mathcal{N}_{\sigma^{(2)}}^{\Sigma_K}$$

Proof of the claim:

Given $\psi \in C^{0,1}(\partial E)$, with supp$\psi \subset \Sigma_K$ and $\int_{\partial E} \psi = 0$, consider $u^{(i)}$ solution to the problem:

$$\begin{cases} \text{div}(\sigma^{(i)} \nabla u^{(i)}) = 0, & \text{in} \quad E \\ \sigma^{(i)} \nabla u^{(i)} \cdot \nu = \psi, & \text{on} \quad \partial E, \qquad i = 1, 2. \end{cases}$$

We want to show that:

$$\left\langle \psi, \left(\mathcal{N}_{\sigma^{(1)}}^{\Sigma_K} - \mathcal{N}_{\sigma^{(2)}}^{\Sigma_K}\right) \psi \right\rangle = \int_E \left(\sigma^{(2)}(x) - \sigma^{(1)}(x)\right) \nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x) dx$$
$$= 0.$$

- Extend the solution $u^{(i)}$ with a bounded extension operator:

$$T : H^{\frac{1}{2}}(\partial E \cap \Omega) \longrightarrow H^{1}(\Omega),$$

such that, given $f \in H^{\frac{1}{2}}(\partial E \cap \Omega)$, we have

$$Tf\big|_{\Sigma_1} = 0.$$

- Denote

$$\overline{u}^{(i)} = \begin{cases} u^{(i)}, & \text{in} \quad E \\ T\left(u^{(i)}\big|_{\partial E \cap \Omega}\right), & \text{in} \quad D \end{cases} \in H^{1}(\Omega).$$

- Consider an augmented domain $\Omega_0 = \Omega \cup D_0$
- Extend $\sigma^{(i)}$ to $\widetilde{\sigma}^{(i)}$ such that $\widetilde{\sigma}^{(i)} = I$ on D_0, $i = 1, 2$
- For $y \in \Omega_0$ define the modified Neumann kernel $\widetilde{N}^{(i)}$:

$$\begin{cases} \mathrm{div}(\widetilde{\sigma}^{(i)} \widetilde{N}^{(i)}(\cdot, y)) = -\delta(x - y), & \text{in} \quad \Omega_0 \\ \widetilde{\sigma}^{(i)} \nabla \widetilde{N}^{(i)} \cdot \nu = 0, & \text{on} \quad \partial\Omega_0 \cap \partial\Omega \\ \widetilde{\sigma}^{(i)} \nabla \widetilde{N}^{(i)} \cdot \nu = -\frac{1}{|\partial\Omega_0 \setminus \bar{\Omega}|}, & \text{on} \quad \partial\Omega_0 \setminus \bar{\Omega}. \end{cases}$$

- For $x \in E$ we have

$$\begin{aligned} u^{(i)}(x) &= -\int_{\Omega} \overline{u}^{(i)}(y) \,\mathrm{div}_y \left(\sigma^{(i)}(y) \nabla_y \widetilde{N}^{(i)}(y, x) \right) \, dy \\ &= \int_{\Sigma_K} \psi \widetilde{N}^{(i)}(y, x) \, dS(y) \\ &+ \int_{D} \sigma^{(i)}(y) \nabla_y \overline{u}^{(i)}(y) \cdot \nabla_y \widetilde{N}^{(i)}(y, x) \, dy. \end{aligned}$$

$$\nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x)$$

$$= \int_{\Sigma_K \times \Sigma_K} \psi(y)\psi(z)\nabla_x \widetilde{N}^{(1)}(y,x) \cdot \nabla_x \widetilde{N}^{(2)}(z,x)\, dydz$$

$$+ \int_{\Sigma_K \times D} \psi(y)\sigma_{lk}^{(2)}(z)\partial_{z_l}\overline{u}^{(2)}(z)\partial_{z_k}\Big(\nabla_x \widetilde{N}^{(1)}(y,x) \cdot \nabla_x \widetilde{N}^{(2)}(z,x)\Big)\ dydz$$

$$+ \int_{D \times \Sigma_K} \psi(z)\sigma_{lk}^{(1)}(y)\partial_{y_l}\overline{u}^{(1)}(z)\partial_{y_k}\Big(\nabla_x \widetilde{N}^{(2)}(z,x) \cdot \nabla_x \widetilde{N}^{(1)}(y,x)\Big)\ dydz$$

$$+ \int_{D \times D} \sigma_{lk}^{(2)}(z)\partial_{z_l}\overline{u}^{(2)}(z)\sigma_{nm}^{(1)}(y)\partial_{y_n}\overline{u}^{(1)}(z)$$

$$\times \partial_{z_k}\partial_{y_m}\left(\nabla_x \widetilde{N}^{(2)}(z,x) \cdot \nabla_x \widetilde{N}^{(1)}(y,x)\right)\ dydz.$$

- Define for $y, z \in D \cup D_0$

$$S(y,z) = \int_E \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x \widetilde{N}^{(1)}(y,x) \cdot \nabla_x \widetilde{N}^{(2)}(z,x) dx.$$

- For any $y, z \in \left(\overline{D} \cup \overline{D}_0\right)^{\circ}$

$$\begin{aligned} \text{div}_y \left(\sigma^{(1)}(y) \nabla_y S(y,z)\right) &= 0, \\ \text{div}_z \left(\sigma^{(2)}(z) \nabla_z S(y,z)\right) &= 0, \end{aligned}$$

- $\sigma^{(1)} = \sigma^{(2)}$ on $D \Rightarrow$

$$S(y,z) = \int_{\Omega} \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x \widetilde{N}^{(1)}(y,x) \cdot \nabla_x \widetilde{N}^{(2)}(z,x) dx$$

- For $y, z \in D_0$ "far" from $\partial\Omega$, we obtain:

$$S(y,z) = \left\langle \sigma^{(1)} \nabla \widetilde{N}^{(1)}(y,\cdot) \cdot \nu, \left(\mathcal{N}^{\Sigma_1}_{\sigma^{(2)}} - \mathcal{N}^{\Sigma_1}_{\sigma^{(1)}}\right) \sigma^{(2)} \nabla \widetilde{N}^{(2)}(z,\cdot) \cdot \nu \right\rangle$$
$$= 0.$$

- By the $C^{1,\alpha}$ regularity of the interfaces Σ_j within D, $S(y,z)$ satisfies the unique continuation property in each variable $y, z \in \left(\overline{D} \cup \overline{D}_0\right)^{\circ}$ hence

$$S(y,z) = 0, \qquad \text{for any} \quad y, z \in D.$$

- $$\begin{aligned}
&\int_E \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x)\, dx \\
&= \int_{\Sigma_K \times \Sigma_K} \psi(y)\psi(z) S(y,z)\, dy\, dz \\
&+ \int_{\Sigma_K \times D} \psi(y)\sigma^{(2)}_{lk}(z)\partial_{z_l}\overline{u}^{(2)}(z)\partial_{z_k} S(y,z)\, dy\, dz \\
&+ \int_{D \times \Sigma_K} \psi(z)\sigma^{(1)}_{lk}(y)\partial_{y_l}\overline{u}^{(1)}(z)\partial_{y_k} S(y,z)\, dy\, dz \\
&+ \int_{D \times D} \sigma^{(2)}_{lk}(z)\partial_{z_l}\overline{u}^{(2)}(z)\sigma^{(1)}_{nm}(y)\partial_{y_n}\overline{u}^{(1)}(z)\partial_{z_k}\partial_{y_m} S(y,z)\, dy\, dz \\
&= 0.
\end{aligned}$$

- $$\begin{aligned}
&\left\langle \psi, \left(\mathcal{N}^{\Sigma_K}_{\sigma^{(1)}} - \mathcal{N}^{\Sigma_K}_{\sigma^{(2)}}\right) \psi \right\rangle \\
&= \int_E \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x)\, dx = 0. \qquad \square
\end{aligned}$$

Outline

Modified Tartar's example.

- Take $v = (v', v_n) \in \mathbb{R}^n_+$ and define

$$M = \left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline 0'^T & v_n \end{array} \right),$$

- $x = M\xi$
-

$$I \quad , \quad \sigma = \frac{QQ^T}{\det Q},$$

where $Q = M^{-1}$. σ is the push-forward of the isotropic homogeneous conductivity I through the change of coordinates $x = M\xi$.

$$
\begin{aligned}
g = M^T M &= \left(\begin{array}{c|c} I_{(n-1)} & 0' \\ \hline v'^T & v_n \end{array}\right)\left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline 0'^T & v_n \end{array}\right) \\
&= \left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline v'^T & |v'|^2 + v_n^2 \end{array}\right).
\end{aligned}
$$

Therefore

$$
g_{(n-1)} = I_{(n-1)},
$$

for any choice of $v \in \mathbb{R}^n_+$.

The entire family of anisotropic conductivities:

$$\sigma = \frac{QQ^T}{\det Q} = v_n \left(\begin{array}{c|c} I_{(n-1)} + \frac{1}{v_n^2} v' v'^T & -\frac{1}{v_n^2} v' \\ \hline -\frac{1}{v_n^2} v'^T & \frac{1}{v_n^2} \end{array} \right)$$

is s.t:

$$N_\sigma(x', y') = N_I(x', y'), \qquad \forall\, x', y' \in \Pi_n.$$

That is, any such σ is undistinguishable from the identity I when the corresponding N-D map (or D-N map) on Π_n is given.

Thank you!