STEKLOFF EIGENVALUES IN INVERSE SCATTERING

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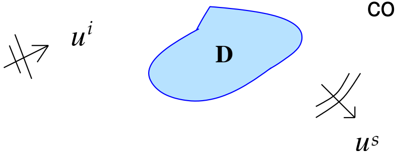
Outline of the talk

- Introduction (c.f. Prof. Cakoni's talk)
- Stekloff eigenvalues for the Helmholtz equation
  - Joint with D. Colton and S. Meng (Delaware) and F. Cakoni (Rutgers)
  - Modified far field operator
  - The far field equation and identifying Stekloff eigenvalues from scattered data
  - Numerical examples
- Stekloff eigenvalues for the Maxwell system
  - Joint with J. Camano (Concepcion, Chile) and C. Lackner (TU Vienna).
  - Existence of Stekloff eigenvalues
  - Modified impedance boundary condition
  - The far field equation and identifying modified Stekloff eigenvalues from scattered data

Scattering by an Inhomogeneous Medium



The direct scattering problem under consideration is to find u such that

$$\Delta u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3$$
$$u = u^s + u^i$$
$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i(x) = \mathrm{e}^{ikd\cdot x}$, $|d| = 1$, $n := n(x)$ is piecewise smooth and $n = 1$ on $\mathbb{R}^3 \setminus \overline{D}$ where D is a smooth bounded domain with connected complement. There exists a unique solution u of the direct scattering problem for which u^s has the asymptotic behavior

$$u^s(x,d) = \frac{e^{ikr}}{r} u_\infty(\hat{x},d) + O\left(\frac{1}{r^2}\right)$$

where $\hat{x} = x/|x|$. The function u_∞ is the far field pattern of u^s.

Target Signatures

Assume $u_\infty(\hat{x}, d)$ is known for all d and $\hat{x}$. We want to find target signatures that can be used to signal changes in D or n, or that can be used to identify D and n from a library of signatures. Here are some examples

- The Singularity Expansion Method attempts to measure scattering resonances from time domain data. See for example Melrose[1] and Baum[2]
- Transmission Eigenvalues (see Prof. Cakoni's talk)
- Stekloff eigenvalues. There is a long history of the use of the related Dirichlet-to-Neumann map in inverse problems.

[1] *Geometric Scattering Theory* by Richard Melrose

[2] Discrimination of buried targets via the singularity expansion, *Inverse Problems* 13 (1997), 557-570.

Transmission Eigenvalues

The far field operator $F : L^2(S^2) \to L^2(S^2)$ where $S^2 := \{x : |x| = 1\}$ is defined by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) ds_d$$

and is injective unless k is a transmission eigenvalue, i.e. a value of k such that there exists a nontrivial solution to

$$\begin{aligned} \Delta w + k^2 n w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D \end{aligned}$$

where ν is the unit outward normal to ∂D.

Transmission eigenvalues can be determined using the operator F and carry information about n [Cakoni, Colton & Haddar 2010].

Transmission Eigenvalues

Let k_1 be the first transmission eigenvalue and suppose $n(x) > 1$ for $x \in \overline{D}$ or $n(x) < 1$ for $x \in \overline{D}$. Then, given k_1 and a knowledge of D, a constant n_0 can be determined such that the scattering problem for $n(x) = n_0$ also has k_1 as its first transmission eigenvalue. Then

$$\min_{\overline{D}} n(x) \leq n_0 \leq \max_{\overline{D}} n(x).$$

[Cakoni, Gintides, & Haddar, 2010]

So changes in transmission eigenvalues could perhaps be used to detect changes in n or the shape of D.

Comments on Transmission Eigenvalues

1. An estimate of the average value of $n(x)$ is determined.
2. Transmission eigenvalues are a physical characterization of the media which corresponds to the non-scattering of special incident fields.
3. To determine transmission eigenvalues require sweeping through frequency.
4. Only real transmission eigenvalues can be detected (so the scatterer needs to be a dielectric).

Stekloff Eigenvalue Problem

We propose an alternative approach to using transmission eigenvalues by modifying the far field operator and make use of Stekloff eigenvalues[3].

Given a bounded domain D, the Stekloff eigenvalue problem is to find $u \neq 0$, $u \in H^1(D)$ and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} \Delta w + k^2 n w &= 0 \quad \text{in } D, \\ \frac{\partial w}{\partial \nu} + \lambda w &= 0 \text{ on } \partial D. \end{aligned}$$

Stekloff eigenvalues will be our proposed target signature, and we need to determine them from far field data.

[3] Stekloff eigenvalues in Inverse Scattering, F. Cakoni, D. Colton, S. Meng and P. Monk, to appear in *SIAM J. Appl. Math.*

An aside about Stekloff eigenvalues

Standard Stekloff problem (1895): Find λ and $u \neq 0$ such that

$$\begin{aligned} \Delta u &= 0 \text{ in } D, \\ \frac{\partial u}{\partial \nu} + \lambda u &= 0 \text{ on } \partial D. \end{aligned}$$

This has been studied extensively

1. Eigenvalues $0 = \lambda_1 > \lambda_2 > \to -\infty$
2. Relation to Dirichlet-to-Neumann map
3. Geometric spectral theory

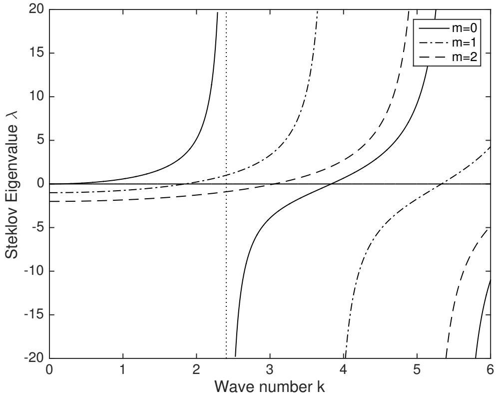
Vladimir Steklov[4]

[4] *The Legacy of Vladimir Andreevich Steklov* by Nikolay Kuznetsov et al., Notices of the AMS, 61 (2014) http://www.ams.org/notices/201401/rnoti-p9.pdf

Stekloff eigenvalues for the unit disk

Stekloff eigenvalues for the Helmholtz equation are less well behaved.

Here are graphs pf the first three Stekloff eigenvalues for the unit disk as a function of k for $n(x) = 1$



Stekloff Eigenvalues

Theorem

Assume that n is real valued. Then Stekloff eigenvalues exist, are real and are discrete.

The case when $n(x)$ is complex valued (i.e. the scattering object is absorbing) is more difficult since then the eigenvalue problem is no longer self-adjoint.

Theorem

Assume that $n(x) = n_1(x) + i\dfrac{n_2(x)}{k}$ *when* $n_1 > 0$ *and* $n_2 > 0$*. Then*

1. *There exist infinitely many Stekloff eigenvalues in the complex plane and they form a discrete set without finite accumulation points.*
2. *Except for a finite number of eigenvalues, all the Stekloff eigenvalues lie in a wedge of arbitrarily small angle with lower edge on the negative x-axis.*

Modified Far Field Operator

To define the modified far field operator let ν be the unit outward normal to ∂D and h denote the solution of the exterior impedance problem:

$$\begin{aligned}
\Delta h + k^2 h = 0, & \qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \\
h(x) = \mathrm{e}^{ikd\cdot x} + h^s(x) & \qquad \text{in } \mathbb{R}^3 \setminus D \\
\frac{\partial h}{\partial \nu} + \lambda h = 0 & \qquad \text{on } \partial D \\
\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) = 0. &
\end{aligned}$$

For $k > 0$ real and $\lambda \in \mathbb{C}$ with $\Im(\lambda) \geq 0$ this problem has a unique solution.

Modified Far Field Operator

We now replace the far field operator F by the modified far field operator $\mathcal{F} : L^2(S^2) \to L^2(S^2)$ defined by

$$(\mathcal{F}g)(\hat{x}) := \int_{S^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]\, g(d)\, ds(d)$$

where h_∞ is the far field pattern of h

Heuristic argument linking Stekloff and $\mathcal{F}$

Suppose $g \in L^2(S)$ is a non-trivial solution of $\mathcal{F}g = 0$ then

$$\int_S [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)] g(d)\, ds(d) = 0.$$

Then define the Herglotz wave function with kernel g by

$$v_g(x) := \int_S \exp(ikx \cdot d)\, g(d)\, ds(d)$$

and note that

$$w_\infty(\hat{x}) := \int_S u_\infty(\hat{x}, d)\, g(d)\, ds(d)$$

is the far field pattern for

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } \mathbb{R}^3 \\ w &= v_g + w^s \text{ in } \mathbb{R}^3 \setminus D \\ \lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial w^s}{\partial r} - ikw^s \right) &= 0 \end{aligned}$$

In the same way

$$v_\infty(\hat{x}) := \int_S h_\infty(\hat{x}, d)\, g(d)\, ds(d)$$

is the far field pattern for v that satisfies

$$\begin{aligned} \Delta v + k^2 v &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{D} \\ v &= v_g + v^s \text{ in } \mathbb{R}^3 \setminus D \\ \frac{\partial v}{\partial \nu} + \lambda v &= 0 \text{ on } \partial D \\ \lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial v^s}{\partial r} - ikv^s \right) &= 0. \end{aligned}$$

Lemma (Rellich)

Suppose D has connected complement and that the far field pattern z_∞ of a scattering solution to the Helmholtz equation $\Delta z + k^2 z = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ is such that $z_\infty(\hat{x}) = 0$ for all $\hat{x} \in S^2$, then $z = 0$ in $\mathbb{R}^3 \setminus D$.

From $\mathcal{F}g = 0$ we have that $w_\infty(\hat{x}) = v_\infty(\hat{x})$. So by Rellich's Lemma $w(x) = v(x)$ in $\mathbb{R}^3 \setminus D$.

Hence w satisfies the boundary value problem

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } D \text{ (from the equation for } w), \\ \frac{\partial w}{\partial \nu} + \lambda w &= 0 \text{ on } \partial D \text{ (from the boundary condition for } v). \end{aligned}$$

So w will be identically zero unless λ is a Stekloff eigenvalue. Then $v_g = 0$ and we conclude $g = 0$.

Modified Far Field Equation

Now for $z \in D$ let

$$\Phi(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$$

and let Φ_∞ be the far field pattern for $\Phi(x,z)$, i.e.

$$\Phi_\infty(\hat{x},z) = \frac{1}{4\pi}e^{-ik\hat{x}\cdot z}, \qquad \hat{x} = \frac{x}{|x|}.$$

Consider the modified equation

$$(\mathcal{F}g)(\hat{x}) = \Phi_\infty(\hat{x},z)$$

where $\mathcal{F}$ is the modified far field operator and where $g \in L^2(S^2)$.

Properties of the Modified Far Field Equation I

Theorem

If λ is not a Stekloff eigenvalue then for every $\epsilon > 0$ there exists $g_\epsilon^z \in L^2(S^2)$ such that for $z \in D$

$$\|\mathcal{F}g_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} < \epsilon.$$

In addition $\|v_{g_\epsilon^z}\|_{L^2(D)}$ is bounded as $\epsilon \to 0$.

If w is the solution to

$$\begin{aligned} \Delta w + k^2 n w &= 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} + \lambda w &= \frac{\partial \Phi(\cdot, z)}{\partial \nu} + \lambda \Phi(\cdot, z) \quad \text{on } \partial D. \end{aligned}$$

then w can be uniquely decomposed as $w = w^i + w^s$ where $w^i \in H^1(D)$ satisfies $\Delta w + k^2 w = 0$ in D and $w^s \in H^1_{loc}(\mathbb{R}^3)$ satisfies the radiation condition. The function g_ϵ^z is such that $v_{g_\epsilon^z}$ satisfies

$$\|w^i - v_{g_\epsilon^z}\|_{H^1(D)} = O(\epsilon).$$

Properties of the Modified Far Field Operator II

Theorem

1. Assume that λ is not a Stekloff eigenvalue and let g_ϵ^z satisfy the far field equation with error ϵ. Then if $v_{g_\epsilon^z}$ is the Herglotz wave function with kernel g_ϵ^z for every $z \in D$ we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ is bounded as $\epsilon \to 0$.

2. Assume that λ is a Stekloff eigenvalue and let $g_\epsilon^z \in L^2(S^2)$ satisfy the far field equation with error ϵ. Then for all $z \in D$, except for possibly a nowhere dense subset, we have that $\|v_{g_\epsilon^z}\|_{H^1(D)}$ cannot be bounded as $\epsilon \to 0$.

Remark If λ is a Stekloff eigenvalue then we can obtain an approximate solution of the far field equation unless the Stekloff eigenfunction can be uniquely continued as a solution of $\Delta w + k^2 n(x) w = 0$ into all of $\mathbb{R}^3$.

Stekloff Eigenvalues and Nondestructive Testing

In nondestructive testing one is interested in small changes in the inhomogeneity $n(x)$. In particular, suppose $n(x)$ is perturbed by δn giving rise to a change in the Stekloff eigenfunction $w \in H^1(D)$ by δw and Stekloff eigenvalue by $\delta\lambda$. Define

$$(f, g) := \int_D f\,\overline{g}\,dx, \qquad \langle f, g\rangle := \int_{\partial D} f\,\overline{g}\,ds.$$

Then, neglecting quadratic terms, we have that

$$\delta\lambda \approx \frac{k^2\,(\delta n w, w)}{\langle w, w\rangle}.$$

Comments on Stekloff Eigenvalues

The advantage of using Stekloff eigenvalues as a target signature are the following:

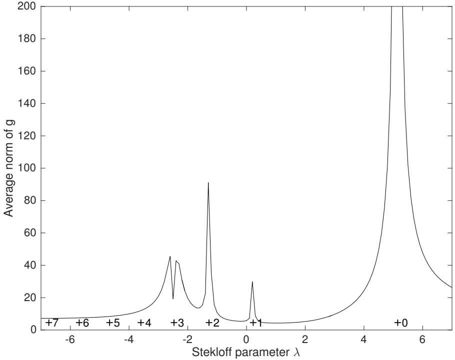
1. The interrogation frequency can be chosen arbitrarily.
2. Stekloff eigenvalues can be computed from far field data for both absorbing and non-absorbing media.

Remarks on Numerical Examples

- The perturbation formula suggests that some eigenvalues are more susceptible to changes in n than others.
- We can usually approximate two or three Stekloff eigenvalues using far field data for the wave numbers and coefficients n that we have used.
- Examples are given using far field data. However, near field data can also be used just as easily.

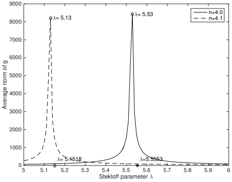
Stekloff Eigenvalues: Unit Disk

- D is the unit disk
- $n(x) = 4$, $k = 1$
- Arbitrary 51 incoming waves
- No extra noise on the data
- Eigenvalues are exact and shown by $+$ in the graph.

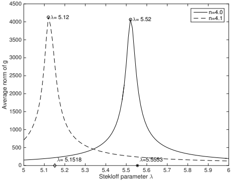


Sensitivity to Noise

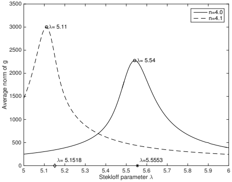
- Same example as before but $n(x) = 4$ or $n(x) = 4.1$
- Noise added pointwise
- Percentage is the relative ℓ^2 norm



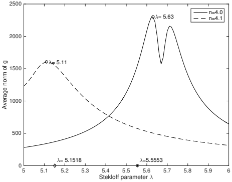
$\epsilon = 0(0\%)$



$\epsilon = 0.01(.57\%)$



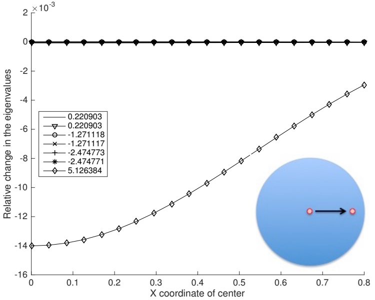
$\epsilon = 0.05(2.9\%)$



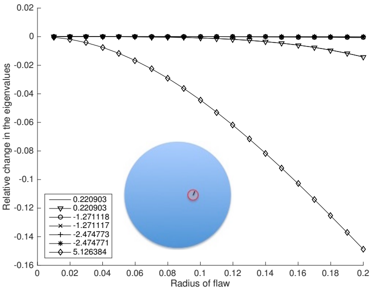
$\epsilon = 0.15(8.6\%)$

Sensitivity of Eigenvalues: Unit Disk with Flaw

The "flaw" is a circular region of radius r_c centered at $(x_c, 0)$ with $n(x) = 1$ inside the flaw. Noise $\epsilon = 0.01$. Wavenumber $k = 1$.



Changing x_c, $r_c = 0.05$

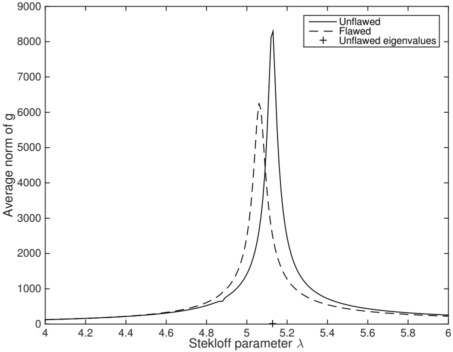


Changing r_c, $x_c = 0.3$.

Plot $(\lambda_{j*}^c - \lambda_j)/|\lambda_j|$, $j = 1, \cdots, 7$ against geometric parameters.

Changes in Eigenvalues: Unit Disk with Flaw

Flaw is radius $r_c = 0.05$ centered at $(0.3, 0)$. All parameters as in the previous examples.



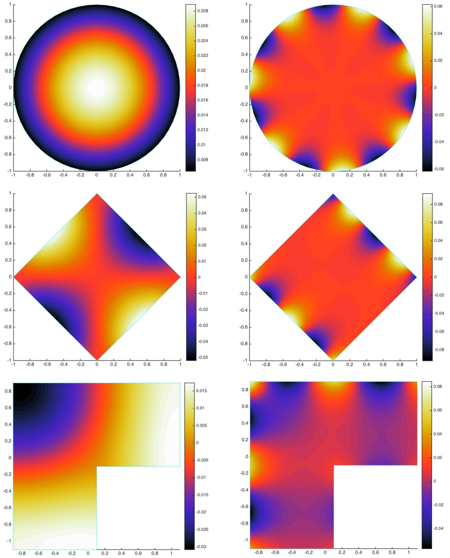
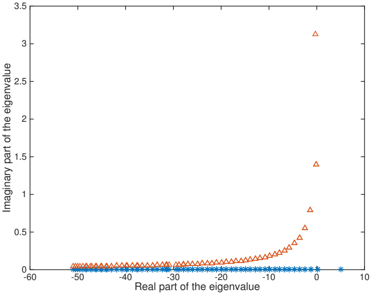


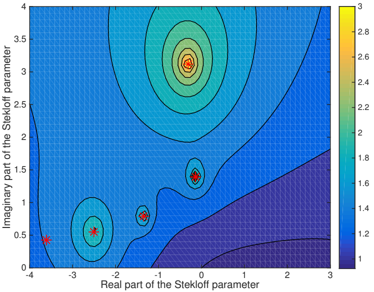
Figure: Density plots of the eigenfunctions for the three domains. Left column: λ_7, Right Column: λ_{14}.

Complex Eigenvalues: Unit Disk $n(x) = 4 + 4i$

Complex eigenvalues can be detected by the same procedure as before but now searching in a region in the complex plane.



Comparison of eigenvalues for $n(x) = 4$ (blue) and $n(x) = 4 + 4i$ (red)



Contours of $\log_{10}(\|g\|)$. Exact Stekloff eigenvalues are shown as $*$.

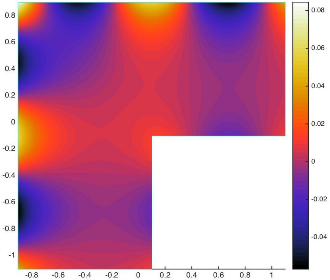
Replacing D by a Ball

The impedance boundary value problem for h can be replaced by an impedance boundary value problem where the boundary condition is prescribed on the boundary of a ball B centered at the origin and containing D in its interior instead of on the boundary of D.

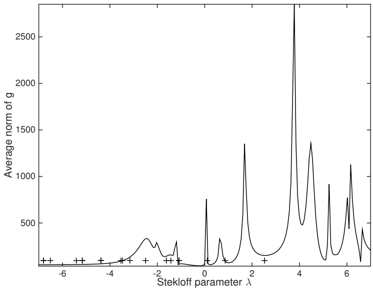
- The price for doing this is that the Stekloff eigenvalues are less sensitive to changes in the refractive index than if the boundary condition in h is prescribed on the boundary of D.
- The advantage of doing this is that for complex geometries the Stekloff eigenvalues perhaps could be more accurately computed from the modified far field equation.

Replacing D by a Ball

D is L-shaped domain with $n(x) = 4$.



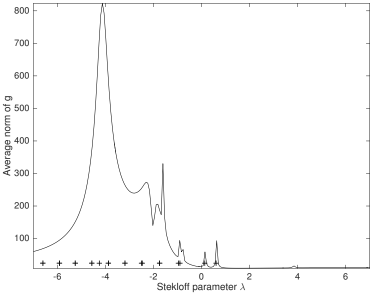
The L-shaped domain



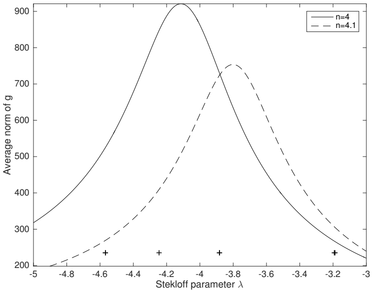
Plot of eigenvalues for h prescribed on ∂D computed using modified far field field operator. Noise $\epsilon = 0$

Replacing D by a Ball

B is a disk of radius 1.5 centered at the origin
D is again L-shaped domain with $n(x) = 4$.



Plot of eigenvalues for h prescribed on ∂B computed using modified far field operator. Noise $\epsilon = 0$



Change in main peak at left when $n(x)$ changes from 4 to 4.1. Noise $\epsilon = 0.01$

The electromagnetic Stekloff problem

We now wish to extend our study for Helmholtz equation to electromagnetism.[5] Let $\mathbf{u}_T = (\boldsymbol{\nu} \times \mathbf{u}) \times \boldsymbol{\nu}$ and define

$$X = \{\mathbf{u} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \mathbf{u}_T \in \mathbf{L}^2_t(\Gamma)\}$$

An obvious analogue of the Helmholtz Stekloff eigenvalue problem is: Find $\mathbf{w} \in X$, $\mathbf{w} \neq 0$ and λ such that

$$\begin{aligned} \nabla\times\nabla\times\mathbf{w} - k^2 n\mathbf{w} &= 0 \text{ in } D \\ \boldsymbol{\nu}\times\nabla\times\mathbf{w} - \lambda\mathbf{w}_T &= 0 \text{ on } \partial D \end{aligned}$$

Note: if $n = 1$ is constant, and λ is a Stekloff eigenvalue, then so is $-k^2/\lambda$. For a sphere eigenvalues can be computed explicitly and the two distinct families verified.

[5] J. Camaño, C. Lackner and P. Monk, Stekloff eigenvalues in electromagnetic scattering, in prep.

A modified Stekloff problem

The Stekloff problem is an artificial problem in our case so we can modify it. Given $\mathbf{f} \in \mathbf{L}^2_t(\Gamma)$ define

$$\mathcal{S}\mathbf{f} = \mathbf{f} - \nabla_\Gamma(\Delta_\Gamma)^{-1}\nabla_\Gamma \cdot \mathbf{f}$$

Then the modified Stekloff eigenvalue problem is to find $\mathbf{w} \in X$, $\mathbf{w} \neq 0$ and λ such that

$$\begin{aligned} \nabla\times\nabla\times\mathbf{w} - k^2 n\mathbf{w} &= 0 \text{ in } D \\ \boldsymbol{\nu}\times\nabla\times\mathbf{w} - \lambda\mathcal{S}\mathbf{w}_T &= 0 \text{ on } \partial D \end{aligned}$$

When n is real this can be cast as an eigenvalue for a compact and self adjoint "Neumann-to-Dirichlet" operator. So existence of a discrete set of modified Stekloff eigenvalues can be verified.

Spliting lemma

Consider the following problem: given $\mathbf{f} \in \boldsymbol{H}^{1/2}(\mathrm{Div}; \partial D)$ find $\mathbf{w}$ such that

$$\begin{array}{rcll} \nabla\times\nabla\times\mathbf{w} - \kappa^2 n\mathbf{w} & = & 0 & \text{in} \quad D\,, \\ \boldsymbol{\nu}\times\nabla\times\mathbf{w} - \lambda\mathcal{S}\mathbf{w}_T & = & \mathbf{f} & \text{on} \quad \partial D\,. \end{array}$$

Lemma

Assume that $\lambda \neq 0$ is not a modified Stekloff eigenvalue. Then the unique solution $\mathbf{w} \in \boldsymbol{H}(\mathrm{curl}, D)$ of the above boundary value problem can be decomposed as $\mathbf{w} = \mathbf{w}^i + \mathbf{w}^s$ where $\mathbf{w}^i \in \boldsymbol{H}(\mathrm{curl}, D)$ solves the Maxwell system in D and $\mathbf{w}^s \in \boldsymbol{H}_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is a radiating field (i.e. satisfies Maxwell's equations with $n = 1$ outside D and the Silver–Müller radiation condition).

By the Stratton-Chu formula, $\mathbf{w} = \mathbf{w}^i + \mathbf{w}^s$ where

$$\begin{aligned} \mathbf{w}^i(\mathbf{x}) &= -\nabla\times\int_{\partial D}\boldsymbol{\nu}(\boldsymbol{y})\times\mathbf{w}(\boldsymbol{y})\Phi(\mathbf{x},\boldsymbol{y})ds_{\boldsymbol{y}} + \nabla\int_{\partial D}\boldsymbol{\nu}(\boldsymbol{y})\cdot\mathbf{w}(\boldsymbol{y})\Phi(\mathbf{x},\boldsymbol{y})ds_{\boldsymbol{y}} \\ &\quad -\int_{\partial B}\boldsymbol{\nu}(\boldsymbol{y})\times\nabla\times\mathbf{w}(\boldsymbol{y})\Phi(\mathbf{x},\boldsymbol{y})ds_{\boldsymbol{y}} \qquad \mathbf{x}\in D, \end{aligned}$$

and

$$\Phi(\mathbf{x},\boldsymbol{z}) := \frac{e^{i\kappa|\mathbf{x}-\boldsymbol{z}|}}{4\pi|\mathbf{x}-\boldsymbol{z}|}.$$

In addition,

$$\mathbf{w}^s(\mathbf{x}) = \kappa^2\int_B (\boldsymbol{n}(\boldsymbol{y})-1)\mathbf{w}(\boldsymbol{y})\Phi(\mathbf{x},\boldsymbol{y})d\boldsymbol{y} - \nabla\int_B \nabla_{\boldsymbol{y}}\cdot(\mathbf{w}(\boldsymbol{y}))\Phi(\mathbf{x},\boldsymbol{y})d\boldsymbol{y} \qquad \mathbf{x}\in\mathbb{R}^3$$

satisfies Maxwell's equations outside D and the Silver–Müller radiation condition.

Properties of the modified Far Field Operator

Since $\mathbf{w}^i$ on the previous slide can be approximated by an electromagnetic Herglotz wave function:

$$\boldsymbol{v}_{\boldsymbol{g}}(\mathbf{x}) := -i\kappa \int_{\mathbb{S}^2} \boldsymbol{g}(\boldsymbol{d})\, e^{-i\kappa \mathbf{x}\cdot\boldsymbol{d}}\, ds_{\boldsymbol{d}}$$

where $\boldsymbol{g} \in \mathbf{L}^2_t(S^2)$, we can now prove the same theorems regarding the electromagnetic Herglotz kernel $\boldsymbol{g}$ as for the Helmholtz kernel g.

This opens the way for an unfortunately incomplete numerical study of electromagnetic Stekloff problems.

Conclusions

- By modifying the far field operator we can find new target signatures for scattering problems.
- We have provided a theoretical study in the case of Stekloff eigenvalues for the Helmholtz and Maxwell systems.
- Extensions to Maxwell's equations with complex n and to standard electromagnetic Stekloff eigenvalues will be undertaken.

Conference on Metamaterials and Applications

A workshop on "*Mathematical Analysis of Metamaterials and Applications*" will be held at the TSIMF(Tsinghua Sanya International Mathematics Forum) located in Sanya, China, during Dec.5-9, 2016.

All local expenses (airport pick up and send off, hotel and meals) will be covered for invited participants. You would just need to cover your own travel to Sanya (a very beautiful seaside city).

More details about the workshop can be found online:

http://ymsc.tsinghua.edu.cn/sanya/2016/MAMAA2016/synopsis_and_organizers.aspx.