Bounds on Stieltjes functions and their applications to fundamental limits of broadband passive cloaking in quasitatic

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A common work with Graeme W. Milton

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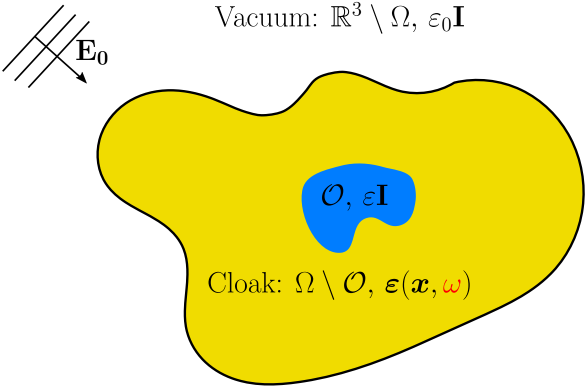
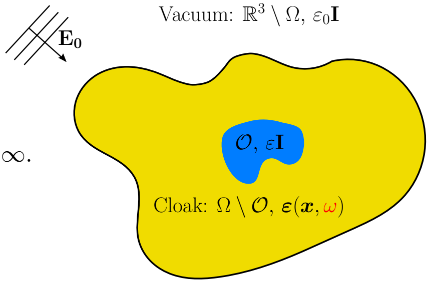
Physical motivation: broadband passive cloaking



Issue: is it possible to construct a passive cloak that will cloak a dielectric inclusion $\mathcal{O}$ over a whole frequency band: $[\omega_-, \omega_+]$?

$\Longrightarrow$ We answer here negatively to these question for the quasistatic regime of Maxwell's equations.

Our model



Quasistatic regime of Maxwell's equations:

$$\begin{cases} \nabla \cdot (\boldsymbol{\varepsilon}(\mathbf{x},\omega)\nabla V(\mathbf{x},\omega)) = 0 \quad on \quad \mathbb{R}^3, \\ V(\mathbf{x},\omega) = -\mathbf{E}_0 \cdot \mathbf{x} + \mathcal{O}(1/|\mathbf{x}|) \quad as \ |\mathbf{x}| \to \infty. \end{cases}$$

where $\mathbf{E}(\mathbf{x},\omega) = -\nabla V(\mathbf{x},\omega)$.

Main correction of the far field of $V(\mathbf{x},\omega)$:

$$V(\mathbf{x},\omega) = -\mathbf{E}_0 \cdot \mathbf{x} + \frac{\mathbf{b}(\omega)\cdot \mathbf{x}}{4\pi|\mathbf{x}|^3} + o(1/|\mathbf{x}|^2).$$

Definition of the polarizability tensor $\boldsymbol{\alpha}(\omega)$

$$\boxed{\mathbf{b}(\omega) = \boldsymbol{\alpha}(\omega)\mathbf{E}_0 = \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0 \mathbf{I})\mathbf{E}(\mathbf{x},\omega)\, \mathrm{d}\mathbf{x}.}$$

Question: Can one construct a passive cloak such that: $\alpha(\omega) = 0$ on $[\omega_-, \omega_+]$?

The cloak: a passive material

Constitutive law of the cloak:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \boldsymbol{\chi}_E \overset{t}{\star} \mathbf{E} \quad \underbrace{\Longrightarrow}_{\mathcal{L}_t:\ \text{Fourier-Laplace transform}} \quad \mathbf{D}(\cdot,\omega) = \boldsymbol{\varepsilon}(\cdot,\omega)\mathbf{E}(\cdot,\omega),$$

where: $\boldsymbol{\varepsilon}(\mathbf{x},\omega) = \varepsilon_0(1 + \mathcal{L}_t(\chi_E)(\mathbf{x},\omega))$.

Passivity of the cloak in the frequency domain:

- (H_1): $\forall \mathbf{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\mathbf{x},\cdot)$ is analytic $\mathbb{C}^+ = \{\omega \in \mathbb{C} \mid \mathrm{Im}(\omega) > 0\}$ and continuous on $\mathrm{cl}\,\mathbb{C}^+$ (causality, $\chi_E(\mathbf{x},\cdot) \in L^1(\mathbb{R})$),
- (H_2): $\forall \mathbf{x} \in \Omega \setminus \mathcal{O}$, $\forall z \in \mathrm{cl}\,\mathbb{C}^+$, $\overline{\boldsymbol{\varepsilon}(\mathbf{x},\omega)} = \boldsymbol{\varepsilon}(\mathbf{x},-\overline{\omega})$ (real fields in the time domain),
- (H_3): $\forall \mathbf{x} \in \Omega \setminus \mathcal{O}, \forall \omega \in \mathbb{R}^+$, $\mathrm{Im}\,\boldsymbol{\varepsilon}(\mathbf{x},\omega) \geq 0$, (passivity: energy balance)
- (H_4): $\forall \mathbf{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\mathbf{x},\omega) \to \varepsilon_0 \mathbf{I}$ as $|\omega| \to \infty$ in $\mathrm{cl}\,\mathbb{C}^+$ $(\chi_E(\mathbf{x},\cdot) \in L^1(\mathbb{R}))$

M. Cessenat (1996),G. Milton (2002), P. Joly's talk this morning $\cdots$

Tools and bibliography

Goal

Derivation of quantitative bounds on the cloaking effect over a frequency band.

Main tools:

Existence of a Stieltjes or/and a Herglotz function associated to the passive linear system in the frequency domain and use of sum rules.

Bibliography:

- Bounds in electromagnetism: G. Milton, D. Eyre and J. Mantese (1997), R. Lipton (2000, 2001, 2004), M. Gustafsson and D. Sjöberg (2010), A. Welters, Y. Avniel, and S. Johnson (2014), O. Miller and al. (2015), $\cdots$, and many others, $\cdots$
- Sum rules: A. Bernland, A. Luger and M. Gustafsson (2010, 2011), $\ldots$
- Cloaking: Steven Johnson and al. (2012), F. Monticone and A. Alu (2014, 2016), A. Norris (2015), $\cdots$

Outline

1. New bounds on Stieltjes functions

2. Applications to our cloaking problem

Herglotz functions

Definition

An analytic function $h : \mathbb{C}^+ \to \mathbb{C}$ is a Herglotz if

$$\operatorname{Im} h(z) \geq 0, \ \forall z \in \mathbb{C}^+.$$

Theorem (Representation)

A necessary and sufficient condition to be a Herglotz function is given by the following representation:

$$h(z) = \alpha\, z + \beta + \int_{\mathbb{R}} \left(\frac{1}{\xi - z} - \frac{\xi}{1+\xi^2} \right) \mathrm{dm}(\xi), \quad \textit{for } \operatorname{Im}(z) > 0,$$

where $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ *and* m *is a positive regular Borel measure for which* $\int_{\mathbb{R}} \mathrm{dm}(\xi)/(1+\xi^2)$ *is finite.*

F. Gesztesy and E. Tsekanovskii (2000), C. Berg (2008), $\cdots$.

Stieltjes functions

Definition

A Stieltjes function is an analytic function $g : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}$ which satisfies:

$$\operatorname{Im} g(z) \leq 0 \ \ \forall z \in \mathbb{C}^+ \quad \text{and} \quad g(x) \geq 0 \ \forall x > 0.$$

Theorem (Representation)

A necessary and sufficient condition to be a Stieltjes function is given by the following representation:

$$g(z) = \alpha + \int_{\mathbb{R}^+} \frac{\mathrm{dm}(\xi)}{\xi + z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}^-,$$

where $\alpha = \lim_{|z| \to +\infty} g(z) \in \mathbb{R}^+$ *and* m *is a positive regular Borel measure, uniquely defined, for which* $\int_{\mathbb{R}^+} \mathrm{dm}(\xi)/(1+\xi)$ *is finite.*

G. A. Baker Jr. and P. R. Graves-Morris (1981), C. Berg (2008), $\cdots$.

Objective: A bound on f over a finite frequency band

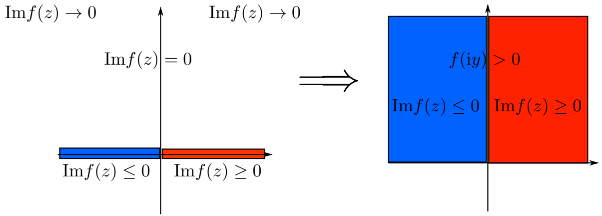
We consider a function f which satisfies the following hypothesis:

- (H_1): f is analytic on $\mathbb{C}^+$ and continuous on $\mathrm{cl}\,\mathbb{C}^+$,
- (H_2): f satisfies $f(-\overline{z}) = \overline{f(z)}, \quad \forall z \in \mathrm{cl}\,\mathbb{C}^+$,
- (H_3): $\mathrm{Im}\, f(z) \geq 0$ for all $z \in \mathbb{R}^+$,
- (H_4): $f(z) \to f_\infty > 0$, when $|z| \to \infty$ in $\mathrm{cl}\,\mathbb{C}^+$.

For instance: $f(z) = \varepsilon(z)$, $f(z) = \mu(z)$, $\cdots$ and we will see that $f(z) = \boldsymbol{\alpha}(z)\mathbf{E}_0 \cdot \mathbf{E}_0$.

$\Longrightarrow$ Construct a Stieltjes and a Herglotz function associated to f.

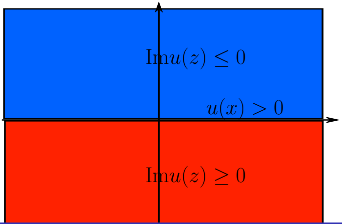
Construction of a Stieltjes function



Definition of the complex square root (branch cut on the positive axis):

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{\mathrm{i} \arg z/2} \text{ if } \arg z \in (0, 2\pi) \text{ and } \sqrt{x} = |x|^{\frac{1}{2}} \text{ if } x \in \mathbb{R}^{+}.$$

We define: $u(z) := f(\sqrt{-z})$.



Construction of a Stieltjes function

Theorem

If a function f satisfy the hypothesis H1-4, then the functions u defined by

$$u(z) := f(\sqrt{-z}), \ \forall z \in \mathbb{C}$$

is a Stieltjes function.

Corollary

The function v defined by

$$v(z) := z\, u(-z) = z f(\sqrt{z}), \ \forall z \in \mathbb{C}$$

is a Herglotz function which is analytic on $\mathbb{C} \setminus \mathbb{R}^{+}$ and negative on $\mathbb{R}^{-*}$. Its associated measure is supported in $\mathbb{R}^{+}$ and its coefficient α is equal to f_{∞}.

Sum-Rules

Proposition (Sum-rules at order 0)

Let h be a Herglotz function which admits the following asymptotic expansions (in any Stolz domain):

$$h(z) = \frac{a_{-1}}{z} + +o\Big(\frac{1}{z}\Big) \quad \text{as } |z| \to 0,$$

and

$$h(z) = \frac{b_{-1}}{z} + o\Big(\frac{1}{z}\Big) \quad \text{as } |z| \to +\infty\,.$$

Then, one have:

$$\lim_{\eta \to 0^+} \lim_{y \to 0^+} \frac{1}{\pi} \int_{\eta < |x| < \eta^{-1}} \operatorname{Im} h(x + \mathrm{i}y)\, \mathrm{d}x = a_{-1} - b_{-1}.$$

A. Bernland, A. Luger and M. Gustafsson (2011)

Objective: To use sum-rules to derive bounds on f over a finite frequency band $[\omega_-, \omega_+]$. More precisely, to genrelasize the approach of A. Bernland, A. Luger and M. Gustafsson (2010,2011) to derive bounds by using the zero order sum rule.

Composition of Herglotz functions

Let us introduce:

$$\mathcal{M}_\Delta = \{\text{probability measures m on } \mathbb{R} \mid \sup(\mathrm{m}) \subset [-\Delta, \Delta]\}.$$

For any $\mathrm{m} \in \mathcal{M}_\Delta$, one defines the Herglotz function:

$$h_{\mathrm{m}}(z) = \int_{-\Delta}^{\Delta} \frac{\mathrm{dm}(\xi)}{\xi - z}, \ \forall z \in \mathbb{C}^+.$$

Thus, $v_{\mathrm{m}} = h_{\mathrm{m}} \circ v$ is also an Herglotz function.

One can easily prove that v_{m} admits the follow assymptotics:

$$v_{\mathrm{m}}(z) = \frac{-\mathrm{m}(\{0\})}{f(0)\, z} + o\left(\frac{1}{z}\right) \text{ as } |z| \to 0 \text{ and } v_{\mathrm{m}}(z) = \frac{-1}{f_\infty\, z} + o\left(\frac{1}{z}\right) \text{ as } |z| \to +\infty.$$

Sum-rules

For any interval $[x_-, x_+] \subset \mathbb{R}^{+*}$, one gets

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_{x_-}^{x_+} \operatorname{Im} v_{\mathrm{m}}(x + \mathrm{i}y)\, \mathrm{d}x \leq \frac{1}{f_\infty} - \frac{\mathrm{m}(\{0\})}{f(0)} \leq \frac{1}{f_\infty}.$$

Optimal bounds

Theorem

Let Δ be a positive real number and $[x_-, x_+]$ be a finite frequency band included in $\mathbb{R}^{+*}$, then one has

$$\sup_{\mathrm{m}\in\mathcal{M}_\Delta} \frac{1}{\pi} \lim_{y\to 0^+} \int_{x_-}^{x_+} \mathrm{Im}\, v_{\mathrm{m}}(x+\mathrm{i}y)\mathrm{d}x = \sup_{\xi\in[-\Delta,+\Delta]} \frac{1}{\pi} \lim_{y\to 0^+} \int_{x_-}^{x_+} \mathrm{Im}\, v_{\delta_\xi}(x+\mathrm{i}y)\mathrm{d}x,$$

where δ_ξ denote the Dirac measure at ξ.

$\Longrightarrow$ For any $\Delta \in \mathbb{R}$, the family of Dirac measures $(\delta_\xi)_{\xi\in\mathbb{R}}$ optimizes the sum-rule on the interval $[x_-, x_+]$ on the set of measures $\mathcal{M}_\Delta$.
For $\mathrm{m} = \delta_\xi$, the sum rule can be rewritten as:

$$\boxed{\lim_{y\to 0^+} \int_{x_-}^{x_+} \mathrm{Im}\left(\frac{1}{\xi - v(x+\mathrm{i}y)}\right) \mathrm{d}x \leq \frac{\pi}{f_\infty}, \ \forall \xi \in \mathbb{R}.}$$

Explicit bound on a transparency window

By definition, in a transparency window $[\omega_-, \omega_+]$

$$\underbrace{\mathrm{Im} f(\omega) = 0}_{\text{physically: no loss}} \ , \forall \omega \in [\omega_-, \omega_+].$$

$\Longrightarrow f$ can extended analytically through the real axis for $\omega \in (\omega_-, \omega_+)$.

Using the family of measures $(\delta_\xi)_{\xi \in \mathbb{R}}$ for the measure m, one revover a bound derived in G. Milton, D. Eyre and J. Mantese (1997):

Proposition (bound in a transparency window)

In a transparency window $[x_-, x_+] = [{\omega_-}^2, {\omega_+}^2]$, the function v satisfies

$$f_\infty(x - x_0) \leq v(x) - v(x_0), \ \forall x, x_0 \in [x_-, x_+] \text{ such that } \ x_0 \leq x,$$

Since $v(z) = z f(\sqrt{z})$, it yields to the following bound on f:

$${\omega_0}^2 (f(\omega_0) - f_\infty) \leq \omega^2 (f(\omega) - f_\infty), \ \forall \omega, \omega_0 \in [\omega_-, \omega_+] \text{ such that } \ \omega_0 \leq \omega.$$

Remark: We proved that this last bound can be also obtained by applying Kramers-Kronig relations on the function f.

The lossy case

By choosing the uniform distribution of $\mathcal{M}_\Delta$ defined by:

$$\mathrm{dm}(\xi) = \frac{\mathbf{1}_{[-\Delta,\Delta]}(\xi)}{2\Delta}\mathrm{d}\xi \quad \text{with} \quad \Delta = \max_{x\in[\omega_-{}^2,\omega_+{}^2]} |v(x)|$$

in the sum rule:

$$\lim_{y\to 0^+} \frac{1}{\pi}\int_{x_-}^{x_+} \operatorname{Im} v_{\mathrm{m}}(x+\mathrm{i}y)\,\mathrm{d}x = \frac{1}{2\Delta}\lim_{y\to 0^+}\int_{x_-}^{x_+} arg\left(\frac{v(x+\mathrm{i}y)-\Delta}{v(x+\mathrm{i}y)+\Delta}\right)\mathrm{d}x \leq \frac{1}{f_\infty},$$

one recovers a bound similar to one derived in

A. Bernland, A. Luger and M. Gustafsson (2011).

Proposition (bound for the lossy case)

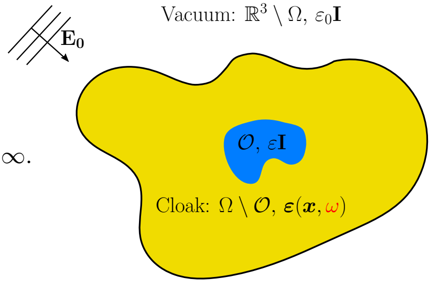
Let $[\omega_-,\omega_+]\subset\mathbb{R}^{+*}$ then the function f satisfies the following inequality:

$$\frac{1}{4}({\omega_+}^2-{\omega_-}^2)f_\infty \leq \max_{x\in[\omega_-,\omega_+]} |\omega^2 f(\omega)|.$$

Remark: One will recover exactly the bound of A. Bernland, A. Luger and M. Gustafsson (2011) by choosing the Herglotz function $v(z) = zf(z)$ instead of $v(z) = zf(\sqrt{z})$.

Recall of the problem and the hypothesis

$$\begin{cases} \nabla \cdot (\boldsymbol{\varepsilon}(\mathbf{x},\omega)\nabla V(\mathbf{x},\omega)) = 0 \quad on \quad \mathbb{R}^3, \\ V(\mathbf{x},\omega) = -\mathbf{E}_0 \cdot \mathbf{x} + \mathcal{O}(1/|\mathbf{x}|) \ \text{ as } \ |\mathbf{x}| \to \infty. \end{cases}$$

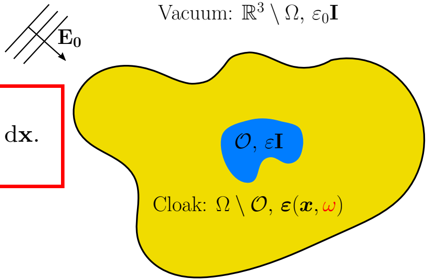


Passivity of the cloak

- $(H1)$: For a.e. $\mathbf{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\mathbf{x},\cdot)$ is analytic $\mathbb{C}^+ = \{\omega \in \mathbb{C} \mid \mathrm{Im}(\omega) > 0\}$ and continuous on $\mathrm{cl}\,\mathbb{C}^+$,
- $(H2)$: For a.e. $\mathbf{x} \in \Omega \setminus \mathcal{O}$, $\forall z \in \mathrm{cl}\,\mathbb{C}^+$, $\overline{\boldsymbol{\varepsilon}(\mathbf{x},\omega)} = \boldsymbol{\varepsilon}(\mathbf{x}, -\overline{\omega})$
- $(H3)$: For a.e. $\mathbf{x} \in \Omega \setminus \mathcal{O}, \forall \omega \in \mathbb{R}^+$, $\mathrm{Im}\,\boldsymbol{\varepsilon}(\mathbf{x},\omega) \geq 0$,
- $(H4)$: For a.e. $\mathbf{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\cdot,\omega) \to \varepsilon_0\mathbf{I}$ as $|\omega| \to \infty$ in $\mathrm{cl}\,\mathbb{C}^+$.

Recall of the problem and the hypothesis

$$\boldsymbol{\alpha}(\omega)\mathbf{E}_0 = \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0 \mathbf{I})\mathbf{E}(\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}.$$



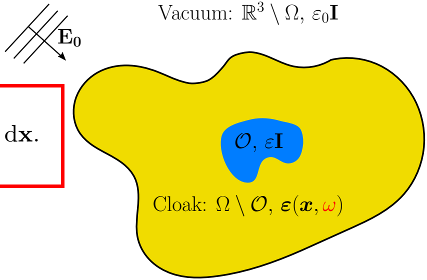
Objective: prove that $\boldsymbol{\alpha}(\cdot)$ satisfy:

- (H1): $\boldsymbol{\alpha}$ is analytic on $\mathbb{C}^+$ and continuous on $\mathrm{cl}\,\mathbb{C}^+$,
- (H2): $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\alpha}(-\overline{\omega}) = \overline{\boldsymbol{\alpha}(\omega)}, \quad \forall \omega \in \mathrm{cl}\,\mathbb{C}^+$,
- (H3): $\mathrm{Im}\,\boldsymbol{\alpha}(\omega) \geq 0$ for all $\omega \in \mathbb{R}^+$,
- (H4): $\boldsymbol{\alpha}(\omega) \to \boldsymbol{\alpha}_\infty > 0$, when $|\omega| \to \infty$ in $\mathbb{C}^+$.

$\Longrightarrow v(\omega) = \omega\,\boldsymbol{\alpha}(\sqrt{\omega})\mathbf{E}_0 \cdot \mathbf{E}_0$ is an Herglotz function $\forall \mathbf{E}_0 \in \mathbb{C}^3$.

Recall of the problem and the hypothesis

$$\boldsymbol{\alpha}(\omega)\mathbf{E}_0 = \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0 \mathbf{I})\mathbf{E}(\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}.$$



Objective: prove that $\boldsymbol{\alpha}(\cdot)$ satisfy:

- (H1): $\boldsymbol{\alpha}$ is analytic on $\mathbb{C}^+$ and continuous on $\mathrm{cl}\,\mathbb{C}^+$,
- (H2): $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\alpha}(-\overline{\omega}) = \overline{\boldsymbol{\alpha}(\omega)}$, $\quad \forall\omega \in \mathrm{cl}\,\mathbb{C}^+$, (from $\overline{\boldsymbol{\varepsilon}(\mathbf{x},\omega)} = \boldsymbol{\varepsilon}(\mathbf{x},-\overline{\omega})$)
- (H3): $\mathrm{Im}\,\boldsymbol{\alpha}(\omega) \geq 0$ for all $\omega \in \mathbb{R}^+$, (energy balance in the harmonic domain)
- (H4): $\boldsymbol{\alpha}(\omega) \to \boldsymbol{\alpha}_\infty > 0$, when $|\omega| \to \infty$ in $\mathbb{C}^+$. (limit behavior of the PDE).

Functional framework

Goal

Prove that the PDE admits a unique solution which depends analytically of ω in $\mathbb{C}^+$ and continuously of ω in $\mathrm{cl}\,\mathbb{C}^+$.

We define the weighted Sobolev space:

$$W_{1,-1}(\mathbb{R}^3) = \{u \in S^{'}(\mathbb{R}^3) \mid (1+|\mathbf{x}|^2)^{-\frac{1}{2}}\, u \in L^2(\mathbb{R}^3) \ \text{ and } \nabla u \in \mathbf{L}^2(\mathbb{R}^3)\}$$

endowed with the Hilbert norm:

$$\|u\|_{W_{1,-1}(\mathbb{R}^3)} = \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Two additional assumptions:

- H5 (Uniformly bounded): $\forall \omega \in \mathrm{cl}\,\mathbb{C}^+$, $\exists \delta > 0$ such that

$$\sup_{\mathbf{x} \in \Omega \setminus \mathcal{O},\, \omega \in \mathrm{cl}\,\mathbb{C}^+} \|\boldsymbol{\varepsilon}(\mathbf{x}, \omega)\| < \infty;$$

- H6 (Coercivity): $\forall \omega \in \mathrm{cl}\,\mathbb{C}^+$, $\exists \gamma(\omega) \in [0, 2\pi($ and $c_2(\omega) > 0$ such that

$$\text{for a.e. } \mathbf{x} \in \Omega \setminus \mathcal{O}, |\mathrm{Im}(e^{i\,\gamma(\omega)} \boldsymbol{\varepsilon}(\mathbf{x}, \omega) u.u)| \geq c_2(\omega)|u|^2 \ \ \forall u \in \mathbb{R}^3.$$

One decomposes $V(\mathbf{x},\omega)$ as $V(\mathbf{x},\omega) = -\mathbf{E}_0 \cdot \mathbf{x} + \underbrace{V_s(\mathbf{x},\omega)}_{W_{1,-1}(\mathbb{R}^3)}$.

To find $V_s(\mathbf{x},\omega) \in W_{1,-1}(\mathbb{R}^3)$ satisfying

$$\begin{cases} \nabla \cdot (\boldsymbol{\varepsilon}(\mathbf{x},\omega)\nabla V_s(\mathbf{x},\omega)) = \nabla \cdot ((\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0 \mathbf{I})\mathbf{E}_0) \text{ on } \mathbb{R}^3 \\ V_s(\mathbf{x},\omega) = \mathcal{O}(1/|\mathbf{x}|) \quad \text{as} \quad |\mathbf{x}| \to \infty. \end{cases}$$

is equivalent to solve the infinite linear system

$$\boxed{\mathbb{A}(\omega)V_s(\omega,\cdot) = \mathbf{f}(\omega).}$$

where $\forall u, v \in W_{1,-1}(\mathbb{R}^3)$:

$$\langle \mathbb{A}(\omega)u, v\rangle = \int_{\mathbb{R}^3} \boldsymbol{\varepsilon}(\mathbf{x},\omega)\nabla u(\mathbf{x})\cdot\overline{\nabla v(\mathbf{x})}\,\mathrm{d}\mathbf{x} \text{ and } \langle \mathbf{f}(\omega), v\rangle = \int_{\mathbb{R}^3} (\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0\mathbf{I})\mathbf{E}_0)\cdot\overline{\nabla v}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

Lemma

For all $\omega \in \overline{\mathbb{C}^+}$, the operator $A(\omega) : W_{1,-1}(\mathbb{R}^3) \mapsto (W_{1,-1}(\mathbb{R}^3))^*$ is an isomorphism. Moreover, $\omega \to A(\omega)$ and $\omega \to A(\omega)^{-1}$ are analytic in $\mathbb{C}^+$.

Theorem

The PDE admits a unique solution $V_s(\cdot,\omega)$ in $W_{1,-1}(\mathbb{R}^3)$ defined by

$$V_s(\cdot,\omega) = A^{-1}(\omega)\mathbf{f}(\omega,\cdot).$$

Thus $\omega \to V_s(\cdot,\omega)$ and $\omega \to \mathbf{E}_s(\cdot,\omega) = -\nabla V_s(\cdot,\omega)$, endowed respectively with the norms: $\|\cdot\|_{W_{1,-1}(\mathbb{R}^3)}$ and $\|\cdot\|_{\mathbf{L}^2(\mathbb{R}^3)}$, are analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}$.

From

$$\boldsymbol{\alpha}(\omega)\mathbf{E}_0 = \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{x},\omega) - \varepsilon_0\mathbf{I})\mathbf{E}(\mathbf{x},\omega)\,\mathrm{d}\mathbf{x}.$$

one finally deduces that

$\boldsymbol{\alpha}$ is analytic on $\mathbb{C}^+$ and continuous on $\operatorname{cl}\mathbb{C}^+$.

and thus:

$v(\omega) = \omega\,\boldsymbol{\alpha}(\sqrt{\omega})\mathbf{E}_0 \cdot \mathbf{E}_0$ is an Herglotz function which satisfies our bounds.

Limits of Passive Cloaking (transparency window)

- On a transparency window where $\mathrm{Im}\boldsymbol{\alpha}(\omega) = 0$ on $[\omega_-, \omega_+]$, we get:

$$ {\omega_0}^2[\boldsymbol{\alpha}(\omega_0) - \boldsymbol{\alpha}_\infty] \leq \omega^2 \, [\boldsymbol{\alpha}(\omega) - \boldsymbol{\alpha}_\infty] $$

$\forall \omega$, $\omega_0 \in [\omega_-, \omega_+]$ such that $\omega \leq \omega_0$.

Remark

This bound is sharp: for a fixed ω_0 such that $\boldsymbol{\alpha}(\omega_0) \leq \boldsymbol{\alpha}_\infty$, the function

$$ \boldsymbol{\alpha}(\omega) = \boldsymbol{\alpha}_\infty - \frac{{\omega_0}^2[\boldsymbol{\alpha}_\infty - \boldsymbol{\alpha}(\omega_0)]}{\omega^2} \quad \text{(Drude Model)} $$

satisfies the equality and the hypothesis $(H1-4)$.

If one can cloak the dielectric inclusion at ω_0: i.e. $\boldsymbol{\alpha}(\omega_0) = 0$, one gets:

$$ \boldsymbol{\alpha}(\omega) \geq \boldsymbol{\alpha}_\infty \frac{\omega^2 - {\omega_0}^2}{\omega^2} \ \text{ if } \ \omega \geq \omega_0 \ \text{ and } \boldsymbol{\alpha}(\omega) \leq \boldsymbol{\alpha}_\infty \frac{{\omega_0}^2 - \omega^2}{\omega^2} \ \text{ if } \ \omega \leq \omega_0 $$

The lossy case

For the lossy case, one has:

$$\frac{1}{4}({\omega_+}^2 - {\omega_-}^2)\,\boldsymbol{\alpha}(\infty)\mathbf{E}_0 \cdot \mathbf{E}_0 \leq \max_{\omega \in [\omega_-, \omega_+]} |\omega^2\, \boldsymbol{\alpha}(\omega)\mathbf{E}_0 \cdot \mathbf{E}_0|.$$

$\Longrightarrow \boldsymbol{\alpha}(\omega)\mathbf{E}_0$ could not approach 0 over the frequency band $[\omega_-, \omega_+]$.

Future work

- Generalize this study for acoustic and full Maxwell's equations. The polarizability tensor is replaced by the forward scattering amplitude.
- Question of broadband passive cloaking for close observer in electromagnetism. Bounds on the DtN map (in progress, joint work with A.Welters and G. W Milton) M. Cassier, A.Welters and G. W Milton, Analyticity of the Dirichlet-to-Neumann map for the time-harmonic Maxwell's equations, *to appear in Extending the theory of composites to other areas of science edited by Graeme W. Milton*

In preparation: M. Cassier and G. W Milton, Bounds on Stieltjes Functions and their Applications to fundamental Limits of Passive Cloaking in the quasistatic regime

Thank you for your attention!