Coercivity of high-frequency scattering problems

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Joint work with: Euan Spence (Bath),
Ilia Kamotski (UCL); Comm Pure Appl Math 2015.

Also with: Ivan Graham (Bath),
Simon Chandler-Wilde (Reading)
Spence, Chandler-Wilde, Graham, S, Comm Pure Appl Math 2011.

Durham, 12 July 2016

Summary

- $\sim$ 1960's - 1980's: tremendous interest in rigorous aspects of scattering/ diffraction:
  - decay at $t \to \infty$ of wave equation $\dfrac{\partial^2 w}{\partial t^2} - c^2 \Delta w = 0$
  - asymptotics as $k \to \infty$ of Helmholtz equation $\Delta u + k^2 u = 0$
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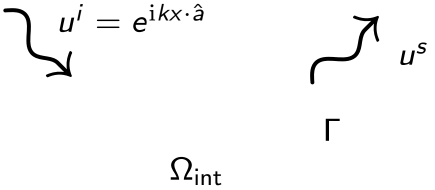
  - asymptotics as $k \to \infty$ of Helmholtz equation $\Delta u + k^2 u = 0$
  - (related in subtle way)
- $\sim$ 2000's - present: interest in Numerical Analysis of Helmholtz for $k \gg 1$
  - e.g. "hybrid asymptotic-numerical methods"
  - This talk: boundary integral equations
  - One analysis question: prove relevant operator is coercive
  - Surprise (?) – appears that for coercivity need *stronger* results than those obtained classically (at least in the context of one classic tool – Morawetz multipliers)

Obstacle scattering problem: acoustically soft/ TE Maxwell (2d) perfectly conducting boundary



$\Delta u^s + k^2 u^s = 0$

in $\Omega_{\text{ext}} := \mathbb{R}^d \setminus \Omega_{\text{int}}$

$u^s + u^i = 0$ on Γ

$k > 0$

Radiation conditions:

$$\frac{\partial u^s}{\partial r} - iku^s = o\left(r^{-\frac{d-1}{2}}\right) \quad \text{as } r \to \infty.$$

$\Longrightarrow$ Uniqueness and existence.

Boundary integral equations

- Green's Integral Representation:

$$u^s(x) = \int_\Gamma \left(\frac{\partial \Phi_k}{\partial n(y)}(x,y) u^s(y) \; - \Phi_k(x,y) \frac{\partial u^s}{\partial n}(y) \right) \, ds(y), \quad x \in \Omega_{\text{ext}}$$

where

$$\Phi_k(x,y) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) & (d=2) \\ \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|} & (d=3) \end{cases}$$

Hence boundary integral equations for $v := \frac{\partial u}{\partial n}$:

- Single layer:

$$S_k v(x) \; := \; \int_\Gamma \Phi_k(x,y) v(y) \, ds(y) \; = \; u^i(x)$$

(uniqueness fails for $k^2 =$ interior Dirichlet eigenvalues)

- (Adjoint) double layer:

$$\left(\frac{1}{2} I + D_k' \right) v(x) := \frac{1}{2} v(x) + \int_\Gamma \frac{\partial \Phi_k}{\partial n(x)}(x,y) v(y) \, ds(y) \; = \; \frac{\partial u^i}{\partial n}(x)$$

(uniqueness fails for $k^2 =$ interior Neumann eigenvalues)

Combined boundary integral equations

Try a *combination* of a double layer and of a single layer:

$$\text{(Double Layer)} \; - i\eta \times \text{(Single Layer)}$$

with a 'coupling constant' $\eta \sim k$ ($k \gg 1$).
I.e. let

$$A_k = \frac{1}{2} I + D_k' - i\,\eta S_k.$$

- $\rightsquigarrow$ 'Combined' boundary integral equation:

$$A_k \left(\frac{\partial u}{\partial n} \right) = f \quad \left(f = \frac{\partial u^i}{\partial n}(x) - i\,\eta\, u^i(x) \right)$$

- At *high frequencies* ($k \gg 1$) kernel of A_k highly oscillatory (and non-linear) in k.

The operator A_k

$$A_k\left(\frac{\partial u}{\partial n}\right) = f$$

- For a fixed k, $\eta > 0$: $A_k : L^2(\Gamma) \to L^2(\Gamma)$ bounded and **invertible** and

$$\partial u/\partial n \in L^2(\Gamma) \text{ if } \Gamma \text{ is Lipschitz (Nečas)}$$

- Q. What do we want to know about A_k?

  1. bound on $\|A_k\|$ (explicit in k)
  2. bound on $\|A_k^{-1}\|$ (explicit in k)
  3. **coercivity:** $\exists\gamma > 0$ such that

$$|(A_k\phi, \phi)_{L^2(\Gamma)}| \geq \gamma\|\phi\|^2_{L^2(\Gamma)}, \quad \forall\phi \in L^2(\Gamma)$$

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     (γ explicit in k) $\longleftarrow$ even harder!

Why is bounding $\|A_k^{-1}\|$ not enough?

$$A_k v = f \qquad \left(v = \frac{\partial u}{\partial n} \right)$$

- Solving numerically using Galerkin method: choose $\mathcal{S}_N \subset L^2(\Gamma)$ (N-dimensional subspace), find $v_N \in \mathcal{S}_N$ such that

$$(A_k v_N, \phi_N)_{L^2(\Gamma)} = (f, \phi_N)_{L^2(\Gamma)}, \quad \forall \phi_N \in \mathcal{S}_N$$

- Want "quasi-optimality": (Lax-Milgramm + Cea's Lemma) :

$$\|v - v_N\|_{L^2(\Gamma)} \leq C(k) \inf_{\phi_N \in \mathcal{S}_N} \|v - \phi_N\|_{L^2(\Gamma)} \qquad (\star)$$

–in some sense "numerical well-posedness"

- $C(k) = \|A_k\|/\gamma_k$ $\therefore$ Bound on $\|A_k^{-1}\|$ can't give $(\star)$ for important $\mathcal{S}_N$

Plan

Multiplier Methods

Multiplier methods

Helmholtz equation

$$\Delta u + k^2 u \quad = 0,$$

where $u(x), \ \ x \in D \subset \mathbb{R}^3, \quad k > 0$

Multiplier methods

Helmholtz equation

$$\int_D \overline{M} \left(\Delta u + k^2 u \right) = 0,$$

where $u(x), \;\; x \in D \subset \mathbb{R}^3, \quad k > 0$

integrate by parts

$$\overline{M} \Delta u = \nabla \cdot (\overline{M} \nabla u) - \nabla \overline{M} \cdot \nabla u$$

get

$$\int_{\partial D} \overline{M} \frac{\partial u}{\partial n} - \int_D \nabla \overline{M} \cdot \nabla u + k^2 \int_D \overline{M}\, u \;\; = 0$$

Some famous (and not so famous) multipliers

$$\int_D \overline{M} \left(\Delta u^s + k^2 u^s \right) = 0,$$

radiation condition for u^s:

$$\therefore \; \frac{\partial u^s}{\partial r} - iku^s = o\left(r^{-\frac{d-1}{2}}\right), \; d = 2, 3$$

$$u^s(x) \sim \frac{e^{\mathrm{i}kr}}{r^{\frac{d-1}{2}}} f(\hat{x}) \quad \text{as} \quad r = |x| \to \infty$$

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- Green (1828), $M = u^s$
- Rellich (1940), e.g. $M = r\frac{\partial u^s}{\partial r} = x \cdot \nabla u^s$
- **Morawetz (1968), e.g.** $M = r\frac{\partial u^s}{\partial r} - ikru^s + \frac{d-1}{2}u^s$

Classic (high frequency) scattering/ diffraction theory

- Enormous interest from 1960's onwards, e.g.,
  - USA – Keller, Lax, Philips, Morawetz (@ Courant), Melrose...
  - USSR/ Russia - Fock, Buslaev, Babich, Vainberg...
- 3 main problems
  1. Wave equation: behaviour as $t \to \infty$
  2. Wave equation: propagation of singularities
  3. Helmholtz: behaviour as $k \to \infty$
  - related in subtle way: "1+2=3" [Vainberg, 1975]

Key concept: (non-)trapping

- as $k \to \infty$ Helmholtz in trapping domains has "almost eigenvalues/eigenfunctions" (resonances)

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- as $k \to \infty$ Helmholtz in trapping domains has "almost eigenvalues/eigenfunctions" (resonances)
- Classic theory can be translated into results about A_k^{-1}.
- Expect that
  1. For Ω_{ext} certain *trapping* domains
  $$\|A_{k_n}^{-1}\| \gtrsim e^{\alpha k_n}, \quad 0 < k_1 < k_2 < \dots \text{ some } \alpha > 0$$
  2. If Ω_{ext} is *non-trapping* then
  $$\|A_k^{-1}\| \lesssim 1, \quad \forall k \geq k_0$$
- Find
  1. Proved
  2. Proved for star-shaped domains [Chandler-Wilde, Monk 2008] using Rellich $\left(M = \dfrac{\partial u}{\partial r}\right)$ (N.B. needed extra work to deal with ∞)

The operator A_k

$$A_k\left(\frac{\partial u}{\partial n}\right) = f$$

- Spaces: $A_k : L^2(\Gamma) \to L^2(\Gamma)$ bounded and invertible and $\partial u/\partial n \in L^2(\Gamma)$ if Γ is Lipschitz
- Q. What do we want to know about A_k?
  1. bound on $\|A_k\|$ (explicit in k) $\longleftarrow$ relatively easy
  2. bound on $\|A_k^{-1}\|$ (explicit in k) $\longleftarrow$ use classic high-frequency scattering theory
  3. coercivity: $\exists\gamma > 0$ such that

$$|(A_k\phi, \phi)_{L^2(\Gamma)}| \geq \gamma\|\phi\|^2_{L^2(\Gamma)}, \quad \forall\phi \in L^2(\Gamma)$$

(γ explicit in k) $\longleftarrow$ why?

Quasi-optimality

$$\|v - v_N\|_{L^2(\Gamma)} \leq C(k) \inf_{\phi_N \in \mathcal{S}_N} \|v - \phi_N\|_{L^2(\Gamma)} \qquad (\star)$$

- Want to establish (with explicit k dependence of C) for
  1. $\mathcal{S}_N$ piecewise polynomials
  2. $\mathcal{S}_{N,k}$ "hybrid" subspace incorporating asymptotics of $v = \dfrac{\partial u^s}{\partial n}$
- For 1. need $N = \mathcal{O}(k^{d-1})$ as $k \to \infty$, possibility of 2. giving $N = \mathcal{O}(1)$.
- k-explicit $(\star)$ for 1. – classic problem, solved by [Melenk, 2011] (needs bound on $\|A_k^{-1}\|$)
- Coercivity (+bound on $\|A_k\|$ – easy) gives $(\star)$ for 1. and 2. k-explicit.

Coercivity

$\exists\gamma > 0$ such that

$$|(A_k\phi, \phi)_{L^2(\Gamma)}| \geq \gamma\|\phi\|^2_{L^2(\Gamma)}, \quad \forall\phi \in L^2(\Gamma)$$

- Not obvious will hold – standard approach to formulations of Helmholtz: prove

$$\text{operator}_k = \text{coercive} + \text{compact}_k$$

- Coercivity for circle (2d) and sphere (3d) $\forall k \geq k_0$, $\gamma = 1$ [Domínguez, Graham, Smyshlyaev, 2007] (Fourier analysis)

Two Coercivity Results using Morawetz Multipliers

Result 1. Ω_{int} Lipschitz star-shaped, a specially constructed "star-combined" $\mathscr{A}_k$ is coercive $\forall k$, $\gamma = \mathcal{O}(1)$. [Spence, Chandler-Wilde, Graham, S., *Comm Pure Appl Math* 2011]

Result 2. Ω_{int} smooth convex, the classical combined A_k is coercive $\forall k \geq k_0$, $\eta > \eta_0 k$, $\gamma = \frac{1}{2} - \varepsilon$ [Spence, I.Kamotski, S. *Comm Pure Appl Math* 2015]

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How?

- A_k arose from $\dfrac{\partial u^s}{\partial n} - \mathrm{i}\eta u^s$ on Γ
- $M = r\dfrac{\partial u^s}{\partial r} - \mathrm{i}kru^s + \dfrac{d-1}{2}u^s \rightsquigarrow$ star-shaped coercivity 1.
- $M = Z(x) \cdot \nabla u^s - \mathrm{i}\eta(x)u^s + \alpha(x)u^s \rightsquigarrow$ smooth convex coercivity 2.

Morawetz - 1

Morawetz & Ludwig (1968): take as a multiplier

$$Mu := \mathbf{x} \cdot \nabla u - \mathrm{i}kru + \frac{d-1}{2} u,$$

Then (the Morawetz-Ludwig identity):

$$2\mathrm{Re}\left(\overline{Mu}(\Delta u + k^2 u)\right) =$$

$$\nabla \cdot \left[2\mathrm{Re}\left(\overline{Mu}\nabla u\right) + \left(k^2|u|^2 - |\nabla u|^2\right)\mathbf{x}\right] - \left(|\nabla u|^2 - |u_r|^2\right) - |u_r - \mathrm{i}ku|^2 .\mathrm{A}$$

corollary:

$$\int_{\partial D} \left[2\mathrm{Re}\left(\overline{Mu}\frac{\partial u}{\partial \nu}\right) + \left(k^2|u|^2 - |\nabla u|^2\right)\mathbf{x} \cdot \nu\right] ds =$$

$$\int_D \left[2\mathrm{Re}\left(\overline{Mu}(\Delta u + k^2 u)\right) + \left(|\nabla u|^2 - |u_r|^2\right) + |u_r - \mathrm{i}ku|^2\right] dx$$

Let Ω_{in} star-shaped $\Leftrightarrow x \cdot n \geq \beta > 0$ ($n = -\nu$). Take $D = \Omega_{ext} \cap B(0, R)$, let $R \to \infty$, use radition conditions. $\Longrightarrow$ k-uniform bounds for D-t-N; error bounds for GO/ GTD, etc.

Morawetz - 1

"Star-combined" BIE (Spence, Chandler-Wilde, Graham, S., *Comm Pure Appl Math 2011*):
In the Morawetz identity, choose $u = S_k\phi$. Take $D = (\Omega_{ext} \cap B(0,R)) \cup \Omega_{int}$, let $R \to \infty$. Then, for

$$\mathcal{A}_k := (\mathbf{x}\cdot\mathbf{n})\left(\frac{1}{2}I + D_k'\right) + \mathbf{x}\cdot\nabla_\Gamma S_k - i\eta S_k, \quad \eta := kr + i\,\frac{d-1}{2}.$$

$$\mathcal{A}_k\left(\frac{\partial u}{\partial n}\right) = f \qquad \left(f = \mathbf{x}\cdot\nabla u^i(x) - i\,\eta(x)\,u^i(x)\right)$$

Coercivity: $\forall k \geq 0$,

$$\mathrm{Re}\,(\mathcal{A}_k\phi, \phi) \geq \gamma\|\phi\|_{L^2}^2, \quad \gamma = \frac{1}{2}\operatorname*{ess\,inf}_{x\in\Gamma}(x\cdot n(x)) > 0.$$

Classical combined (Spence, I. Kamotski, S., *Comm Pure Appl Math 2015)*

Try as a multiplier, with appropriate vector field $\mathbf{Z}(\mathbf{x})$, and scalar functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$:

$$Mu := \mathbf{Z}(\mathbf{x}) \cdot \nabla u - ik\beta(\mathbf{x})u + \alpha(\mathbf{x})u.$$

Then the following Morawetz-type identity holds:

$$2\mathrm{Re}\left(\overline{Mu}\,(\Delta + k^2)u\right) =$$

$$\nabla \cdot \left[2\mathrm{Re}\left(\overline{Mu}\,\nabla u\right) + \left(k^2|u|^2 - |\nabla u|^2\right)\mathbf{Z}\right] +$$

$$\left(2\alpha - \nabla \cdot \mathbf{Z}\right)\left(k^2|u|^2 - |\nabla u|^2\right)$$

$$- 2\ \mathrm{Re}\left(\partial_i Z_j \partial_i u \overline{\partial_j u}\right) - 2\mathrm{Re}\left(\overline{u}\left(\mathrm{i}k\nabla\beta + \nabla\alpha\right) \cdot \nabla u\right)$$

For wave equation Morawetz needed:

- $Z(x), \quad x \in \Omega_{\mathrm{ext}}$
- $\Re\left(\partial_j Z_i\, \xi_i\, \overline{\xi_j}\right) \geq 0,\ \xi \in \mathbb{C}^d,$
- $Z.n > 0$ on Γ,
- $Z(x) \to c\,x$ as $|x| \to \infty$

(almost enough for $\|A_k^{-1}\|$ bound)

have for Ω_{ext} non-trapping in 2-d

For coercivity of A_k we need:

- $Z(x), \quad x \in \Omega_{\mathrm{ext}} \cup \Omega_{\mathrm{int}}$
- $\Re\left(\partial_j Z_i\, \xi_i\, \overline{\xi_j}\right) \geq \theta|\xi|^2,\ \xi \in \mathbb{C}^d,$
- $Z = n$ on Γ,
- $Z(x) \to c\,x$ as $|x| \to \infty$

have for Ω_{int} smooth convex in 2 & 3-d

...non-trapping?

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