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Mathematical and Computational Aspects of Maxwell's Equations,

Space–time Trefftz discontinuous Galerkin methods for wave problems

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Joint work with I. Perugia

Trefftz methods

Consider a PDE $\mathcal{L}u = 0$ that is: (i) linear, (ii) homogeneous (RHS=0), (iii) with piecewise constant coefficients.

Trefftz methods are finite element schemes such that test and trial functions are solutions of the PDE in each element K of the mesh $\mathcal{T}_h$.

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Our main interest is in wave propagation, in:

- Frequency domain, Helmholtz eq. $-\Delta u - k^2 u = 0$
  lot of work done, $h/p/hp$-theory, Maxwell, elasticity...
  (recent survey: Hiptmair, AM, Perugia, arXiv:1506.04521)
- Time domain, wave equation $-\Delta U + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U = 0$

  Trefftz methods are in space–time,
  as opposed to semi-discretisation + time-stepping.

Trefftz methods for wave equation

Why Trefftz methods? Comparing with standard DG,

- better accuracy per DOFs and higher convergence orders;
- PDE properties "known" by discrete space, e.g. dispersion;
- lower dimensional quadrature needed;
- simpler and more flexible;
- adapted bases and (one day) adaptivity...

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Existing works on Trefftz for time-domain wave equation:

- Maciąg, Sokala, Wauer 2005–2011, Liu, Kuo 2016, single element Trefftz;
- Petersen, Farhat, Tezaur, Wang 2009&2014, DG with Lagrange multipliers;
- Egger, Kretzschmar, Schnepp, Tsukerman, Weiland 3×2014–2015, Maxwell equations; Kretzschmar, Moiola, Perugia, Schnepp 2×2015, analysis;
- Banjay, Georgoulis, Lijoka, interior penalty-DG.

Simplest basis: Trefftz polynomials

Consider wave equation $-\Delta U + \frac{1}{c^2} U'' = 0$ in $K \subset \mathbb{R}^{n+1}$ (c const.).

For $\mathbf{d} \in \mathbb{R}^n$, $|\mathbf{d}| = 1$, $f : \mathbb{R} \to \mathbb{R}$ smooth, $f(\mathbf{d} \cdot \mathbf{x} - ct)$ is solution.

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Choose Trefftz space of polynomials of deg. $\leq p$ on element K:

$$\mathbb{T}^p(K) := \left\{ v \in \mathbb{P}^p(K),\ -\Delta v + c^{-2} v'' = 0 \right\}$$
$$= \operatorname{span} \left\{ (\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j,\ {\scriptstyle 0 \leq j \leq p, \atop 1 \leq \ell \leq L(j,n)} \right\},$$ with dimension

$$\dim\left(\mathbb{T}^p(K)\right) = \binom{p+n-1}{n} \frac{2p+n}{p} = \mathcal{O}_{p\to\infty}(p^n) \ll \dim\left(\mathbb{P}^p(K)\right) = \binom{p+n+1}{n+1} = \mathcal{O}_{p\to\infty}(p^{n+1})$$

Taylor polynomial of (smooth) U belongs to $\mathbb{T}^p(K)$.

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Choice of directions $\mathbf{d}_{j,\ell}$: (corresponding to homog. polyn. deg. j)

- $n = 1$, left/right directions $\mathbf{d}_{j,1} = 1$, $\mathbf{d}_{j,2} = -1$, $\mathbb{T}^p(K) = \operatorname{span}\{(x \pm ct)^j\}$;
- $n = 2$, any distinct $\{\mathbf{d}_{j,\ell}\}_{\ell=1,\dots,2j+1}$ give a basis;
- $n = 3$, $(\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j$ linearly indep. $\iff$ $[Y_N^m(\mathbf{d}_{j,\ell})]_{N \leq j, m; \ell}$ full rank.

Initial–boundary value problem

First order initial–boundary value problem (Dirichlet): find $(v, \boldsymbol{\sigma})$

$$\begin{cases} \nabla v + \dfrac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1},\ n \in \mathbb{N}, \\ \nabla \cdot \boldsymbol{\sigma} + \dfrac{1}{c^2}\dfrac{\partial v}{\partial t} = 0 & \text{in } Q, \\ v(\cdot, 0) = v_0, \quad \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega, \\ v(\mathbf{x}, \cdot) = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Equivalent to $-\Delta U + c^{-2}\frac{\partial^2}{\partial t^2} U = 0$ setting $v = \frac{\partial U}{\partial t}$ and $\boldsymbol{\sigma} = -\nabla U$.

Velocity c piecewise constant. $\Omega \subset \mathbb{R}^n$ Lipschitz bounded.

Extensions:

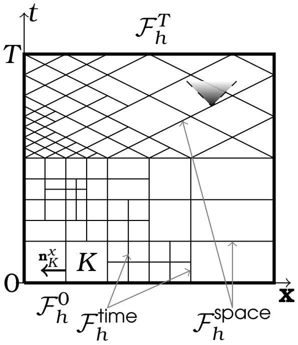
- Neumann $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ & Robin $\frac{\vartheta}{c} v - \boldsymbol{\sigma} \cdot \mathbf{n} = g$ BCs (✓),
- Maxwell equations (✓),
- elasticity,
- 1^{st} order hyperbolic systems ($\sim$),
- Maxwell equations in dispersive materials...

Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on Q.
Assume: $c = c(\mathbf{x})$ constant in elements.

Assume: each face $F = \partial K_1 \cap \partial K_2$ with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c|\mathbf{n}_F^x| < n_F^t$, denote $F \subset \mathcal{F}_h^{\text{space}}$, or
- time-like: $n_F^t = 0$, denote $F \subset \mathcal{F}_h^{\text{time}}$.



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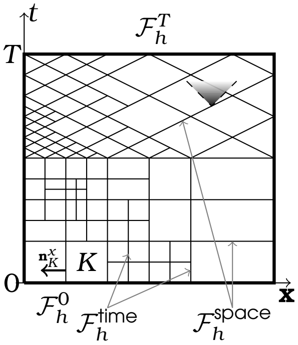

DG notation:

$$\{\!\{w\}\!\} := \frac{w_{|K_1} + w_{|K_2}}{2}, \quad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}_{|K_1} + \boldsymbol{\tau}_{|K_2}}{2},$$

$$[\![w]\!]_{\mathbf{N}} := w_{|K_1}\mathbf{n}_{K_1}^x + w_{|K_2}\mathbf{n}_{K_2}^x,$$

$$[\![\boldsymbol{\tau}]\!]_{\mathbf{N}} := \boldsymbol{\tau}_{|K_1} \cdot \mathbf{n}_{K_1}^x + \boldsymbol{\tau}_{|K_2} \cdot \mathbf{n}_{K_2}^x,$$

$$[\![w]\!]_t := w_{|K_1} n_{K_1}^t + w_{|K_2} n_{K_2}^t = (w^- - w^+) n_F^t,$$

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$$\mathcal{F}_h^0 := \Omega \times \{0\}, \qquad \mathcal{F}_h^T := \Omega \times \{T\},$$

$$\mathcal{F}_h^\partial := \partial\Omega \times [0, T].$$

DG elemental equation and numerical fluxes

Trefftz space:

$$\mathbf{T}(\mathcal{T}_h) := \Big\{(w, \boldsymbol{\tau}) \in L^2(Q), (w|_K, \boldsymbol{\tau}|_K) \in H^1(K)^{1+n},$$

$$\nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\tau} + c^{-2} \frac{\partial w}{\partial t} = 0 \;\forall K \in \mathcal{T}_h\Big\}.$$

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Multiplying PDEs with test $(w, \boldsymbol{\tau})$, integrating by parts in K, using Trefftz property and summing over $K \in \mathcal{T}_h$: $\forall (w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h)$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \left((\upsilon\,\boldsymbol{\tau} + \boldsymbol{\sigma}\, w)\cdot \mathbf{n}_K^x + \left(\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \frac{1}{c^2}\,\upsilon\, w\right) n_K^t \right) \mathrm{d}S = 0.$$

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We approximate skeleton traces of $(v,\boldsymbol{\sigma})$ with numerical fluxes $(\widehat{v}_{hp}, \widehat{\boldsymbol{\sigma}}_{hp})$, defined as $\alpha, \beta \in L^\infty(\mathcal{F}_h^{\text{time}} \cup \mathcal{F}_h^{\partial})$

$$\widehat{v}_{hp} := \begin{cases} v_{hp}^- \\ v_{hp} \\ v_0 \\ \{\!\{v_{hp}\}\!\} + \beta [\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} \\ g \end{cases} \qquad \widehat{\boldsymbol{\sigma}}_{hp} := \begin{cases} \boldsymbol{\sigma}_{hp}^- & \text{on } \mathcal{F}_h^{\text{space}}, \\ \boldsymbol{\sigma}_{hp} & \text{on } \mathcal{F}_h^{T}, \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^{0}, \\ \{\!\{\boldsymbol{\sigma}_{hp}\}\!\} + \alpha [\![v_{hp}]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\text{time}}, \\ \boldsymbol{\sigma}_{hp} - \alpha(v-g)\mathbf{n}_\Omega^x & \text{on } \mathcal{F}_h^{\partial}. \end{cases}$$

$\alpha = \beta = 0 \to$ Kretzschmar–S.–T.–W., $\quad \alpha\beta \geq \frac{1}{4} \to$ Monk–Richter.

TDG formulation

Substituting the fluxes in the elemental equation and choosing any finite-dimensional $\mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$, write TDG as:

Seek $(v_{hp}, \boldsymbol{\sigma}_{hp}) \in \mathbf{V}_p(\mathcal{T}_h)$ s.t., $\quad \forall (w, \boldsymbol{\tau}) \in \mathbf{V}_p(\mathcal{T}_h)$,

$\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) = \ell(w, \boldsymbol{\tau})$ where

$$
\begin{aligned}
\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) :=& \int_{\mathcal{F}_h^{\text{space}}} \Big(\frac{v_{hp}^{-} [\![w]\!]_t}{c^2} + \boldsymbol{\sigma}_{hp}^{-} \cdot [\![\boldsymbol{\tau}]\!]_t + v_{hp}^{-} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \boldsymbol{\sigma}_{hp}^{-} \cdot [\![w]\!]_{\mathbf{N}} \Big) \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{\text{time}}} \Big(\{\!\{ v_{hp} \}\!\} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \{\!\{ \boldsymbol{\sigma}_{hp} \}\!\} \cdot [\![w]\!]_{\mathbf{N}} + \alpha [\![v_{hp}]\!]_{\mathbf{N}} \cdot [\![w]\!]_{\mathbf{N}} + \beta [\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \Big) \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{T}} (c^{-2} v_{hp} w + \boldsymbol{\sigma}_{hp} \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} (\boldsymbol{\sigma}_{hp} \cdot \mathbf{n}_\Omega + \alpha v_{hp}) \, w \, \mathrm{d}S, \\
\ell(w, \boldsymbol{\tau}) :=& \int_{\mathcal{F}_h^{0}} (c^{-2} v_0 w + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} g (\alpha w - \boldsymbol{\tau} \cdot \mathbf{n}_\Omega) \, \mathrm{d}S.
\end{aligned}
$$

Global, implicit and explicit schemes

1. Trefftz-DG formulation is global in space–time domain Q: huge linear system!   Might be good for adaptivity.

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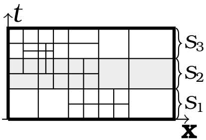
2 If mesh is partitioned in time-slabs $\Omega \times (t_{j-1}, t_j)$, matrix is block lower-triangular: for each time-slab a system can be solved sequentially: implicit method.



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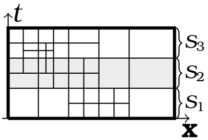
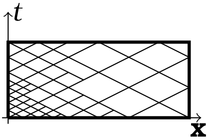


3 If mesh is suitably chosen, Trefftz-DG solution can be computed with a sequence of local systems: explicit method, allows parallelism!

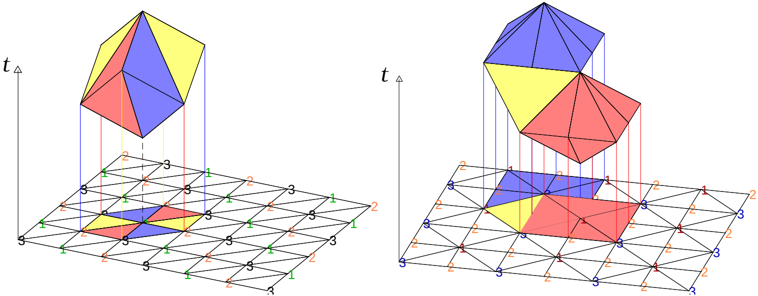


"Tent pitching algorithm" of Üngör–Sheffer, Monk–Richter, Gopalakrishnan–Monk–Sepúlveda, Gopalakrishnan–Schöberl–Wintersteiger...

Versions 1–2–3 are algebraically equivalent (on the same mesh).

Tent-pitched elements

Tent-pitched elements/patches obtained from regular space meshes in 2+1D give parallelepipeds or octahedra+tetrahedra:



Trefftz requires quadrature on faces only:
only the shape of space elements matters.
Simplices around a tent pole can be merged in single element.

TDG a priori error analysis

Using jumps and averages, define 2 mesh- and flux-dependent seminorms $|||\cdot|||_{DG} \leq |||\cdot|||_{DG^+}$ on $H^1(\mathcal{T}_h)^{1+n}$, norms on $\mathbf{T}(\mathcal{T}_h)$.

$\forall(v,\boldsymbol{\sigma}),(w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h)$: $(\alpha, \beta > 0)$

$\mathcal{A}(v,\boldsymbol{\sigma};v,\boldsymbol{\sigma}) \geq |||(v,\boldsymbol{\sigma})|||_{DG}^2$ coercivity,

$|\mathcal{A}(v,\boldsymbol{\sigma};w,\boldsymbol{\tau})| \leq 2\,|||(v,\boldsymbol{\sigma})|||_{DG^+}\;|||(w,\boldsymbol{\tau})|||_{DG}$ continuity,

$\Downarrow$

Existence & uniqueness of discrete solution (only for Trefftz!)

Stability and quasi-optimality:

$$|||(v - v_{hp}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp})|||_{DG} \leq 3 \inf_{(w_{hp},\boldsymbol{\tau}_{hp})\in\mathbf{V}_p(\mathcal{T}_h)} |||(v - w_{hp}, \boldsymbol{\sigma} - \boldsymbol{\tau}_{hp})|||_{DG^+}.$$

Energy dissipation: (if $g = 0$)

$$\frac{1}{2}\int_{\Omega\times\{T\}}(c^{-2}v_{hp}^2 + |\boldsymbol{\sigma}_{hp}|^2)\,\mathrm{d}\mathbf{x} \leq \frac{1}{2}\int_{\Omega\times\{0\}}(c^{-2}v_0^2 + |\boldsymbol{\sigma}_0|^2)\,\mathrm{d}\mathbf{x}.$$

Stability and error bound in $L^2(Q)$ norm

Error bound in space–time $L^2(Q)$ norm follows if we have

$$\left\|\frac{w}{c}\right\|_{L^2(Q)} + \|\boldsymbol{\tau}\|_{L^2(Q)^n} \le C_{(\mathcal{T}_h,\alpha,\beta)} |||(w,\boldsymbol{\tau})|||_{DG} \quad \forall (w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h).$$

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This follows from stability of auxiliary inhomogeneous IBVP

$$\begin{cases} \nabla z + \partial\boldsymbol{\zeta}/\partial t = \boldsymbol{\Phi} & \text{in } Q, \quad \boldsymbol{\Phi} \in L^2(Q)^n, \\ \nabla\cdot\boldsymbol{\zeta} + c^{-2}\,\partial z/\partial t = \psi & \text{in } Q, \quad \psi \in L^2(Q), \\ z(\cdot,0) = 0, \quad \boldsymbol{\zeta}(\cdot,0) = \mathbf{0} & \text{on } \Omega, \\ z(\mathbf{x},\cdot) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

$$2\left\|n_t^{\frac{1}{2}}\frac{z}{c}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)} + 2\left\|n_t^{\frac{1}{2}}\boldsymbol{\zeta}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)^n} + \left\|\frac{z}{\beta^{\frac{1}{2}}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})} + \left\|\frac{\boldsymbol{\zeta}\cdot\mathbf{n}_K^x}{\alpha^{\frac{1}{2}}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}}\cup\mathcal{F}_h^{\partial})}$$
$$\leq C^2_{(\mathcal{T}_h,\alpha,\beta)}\left(\|\boldsymbol{\Phi}\|^2_{L^2(Q)^n} + \|c\psi\|^2_{L^2(Q)}\right) \qquad \forall(\boldsymbol{\Phi},\psi) \in L^2(Q)^{n+1}.$$

Stability and error bound in $L^2(Q)$ norm

Error bound in space–time $L^2(Q)$ norm follows if we have

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This holds under further assumptions on mesh and BCs, otherwise we prove stability in weaker mesh-independent norm.

Convergence bounds: hp in 1+1D, h in n+1D

We prove fully-explicit hp best-approximation bounds in 1+1D.

Combined with quasi-optimality $\rightarrow$ convergence bounds:

$$|||(v - v_{hp}, \sigma - \sigma_{hp})|||_{DG} \leq 87 \sum_{K \in \mathcal{T}_h} \frac{(2h_K)^{s_K + \frac{3}{2}}}{p_K^{s_K}} \left|(c^{-1} v, \sigma)\right|_{W_c^{s_K+1,\infty}(K)}$$

with $K = (x_K, x_K + h_K) \times (t_K, t_K + h_K/c)$, $\alpha^{-1} = \beta = c$, $1 \leq s_K \leq p_K$

- Exponential convergence for analytic solutions:
  $\sim \exp(-b\#\mathrm{DOFs})$ instead of $\exp(-b\sqrt{\#\mathrm{DOFs}})$.

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For $n > 1$, approximation in p is hard, in h follows from Taylor/BH:

$$|||(v-v_{hp}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp})|||_{DG} \le \sum_{K\in\mathcal{T}_h} \frac{8(n+2)}{\rho_K^{1+n/2}} \frac{\big((n+1)h_K\big)^{s_K+\frac{1}{2}}}{(s_K-1)!} \left|(c^{-1/2}v, c^{1/2}\boldsymbol{\sigma})\right|_{H_c^{s_K+1}(K)}$$

$\rho_K =$ "chunkiness", $\alpha^{-1} = \beta = c$, $1 \le s_K \le p_K$, (Cartesian mesh).

Two polynomial Trefftz spaces

If $n \geq 2$, not all solutions $(v, \boldsymbol{\sigma})$ of $\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}$, $\nabla \cdot \boldsymbol{\sigma} + \frac{1}{c^2}\frac{\partial v}{\partial t} = 0$ satisfy $(v, \boldsymbol{\sigma}) = (\frac{\partial}{\partial t}U, -\nabla U)$ for U solution of $\Delta U - c^{-2}\frac{\partial^2 U}{\partial t^2} = 0$ (e.g. if $\operatorname{curl} \boldsymbol{\sigma}_0 \neq \mathbf{0}$).

1. So, if we approximate 1^{st} order IBVP coming from a 2^{nd} order one, we use as basis $(\frac{\partial}{\partial t}, -\nabla)(\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j$, as before.

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2 Otherwise, we generate basis by "evolving" polynomial initial conditions. Elements are in the form

$$(v, \boldsymbol{\sigma}) = \sum_{\substack{k \in \mathbb{N}_0, \boldsymbol{\alpha} \in \mathbb{N}_0^n \\ k + |\boldsymbol{\alpha}| \leq p}} \Big(a_{v,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k,\ a_{\sigma_1,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k,\ \ldots,\ a_{\sigma_n,k,\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} t^k \Big),$$

for $a_{v,k,\boldsymbol{\alpha}}, a_{\sigma_1,k,\boldsymbol{\alpha}}, \ldots, a_{\sigma_n,k,\boldsymbol{\alpha}} \in \mathbb{R}$ satisfying recurrence relations

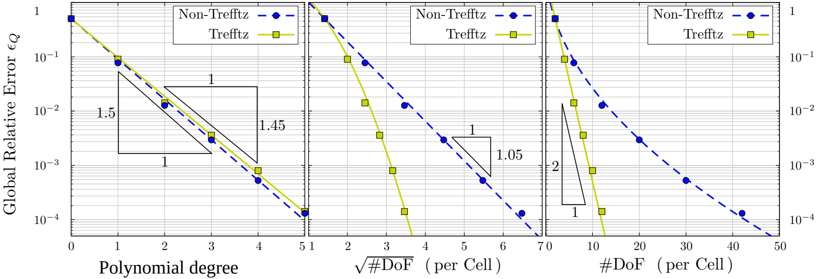
$$a_{v,k,\boldsymbol{\alpha}} = -\frac{c^2}{k} \sum_{m=1}^{n} (\alpha_m + 1) a_{\sigma_m, k-1, \boldsymbol{\alpha}+\mathbf{e}_m}, \qquad k = 1, \ldots, p,$$

$$a_{\sigma_m,k,\boldsymbol{\alpha}} = -\frac{1}{k} (\alpha_m + 1) a_{v, k-1, \boldsymbol{\alpha}+\mathbf{e}_m}, \qquad |\boldsymbol{\alpha}| \leq p - k - 1.$$

2 different discrete spaces, same orders of approximation in h.

Numerical example

Gaussian wave, uniform mesh of squares, p-convergence:



Very weak dependence on flux parameters, even for $\alpha, \beta = 0$.

Maxwell's equations

$$\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} = \mathbf{0}, \qquad \nabla \times \mathbf{H} - \frac{\partial(\epsilon \mathbf{E})}{\partial t} = \mathbf{0} \quad \text{in } Q \subset \mathbb{R}^{3+1},$$

$$\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E} = \mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}(\mathbf{x}, t) \qquad \text{Dirichlet/PEC BCs,}$$

$$\begin{cases} [\![\mathbf{v}]\!]_t := (\mathbf{v}^- - \mathbf{v}^+) \\ [\![\mathbf{v}]\!]_{\mathbf{T}} := \mathbf{n}_{K_1}^{\mathbf{x}} \times \mathbf{v}_{|_{K_1}} + \mathbf{n}_{K_2}^{\mathbf{x}} \times \mathbf{v}_{|_{K_2}} \end{cases} \qquad \text{(tangential) jumps.}$$

Trefftz-DG formulation:

$$\mathcal{A}_{\mathcal{M}}(\mathbf{E}_{hp}, \mathbf{H}_{hp}; \mathbf{v}, \mathbf{w}) = \int_{\mathcal{F}_h^{\text{space}}} \Big(\epsilon \mathbf{E}_{hp}^- \cdot [\![\mathbf{v}]\!]_t + \mu \mathbf{H}_{hp}^- \cdot [\![\mathbf{w}]\!]_t - \mathbf{E}_{hp}^- \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \mathbf{H}_{hp}^- \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} \Big) \,\mathrm{d}S$$

$$+ \int_{\mathcal{F}_h^T} (\epsilon \mathbf{E}_{hp} \cdot \mathbf{v} + \mu \mathbf{H}_{hp} \cdot \mathbf{w}) \,\mathrm{d}S + \int_{\mathcal{F}_h^\partial} \big(\mathbf{H}_{hp} + \alpha(\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E}_{hp})\big) \cdot (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v}) \,\mathrm{d}S$$

$$+ \int_{\mathcal{F}_h^{\text{time}}} \Big(- \{\!\{\mathbf{E}_{hp}\}\!\} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \{\!\{\mathbf{H}_{hp}\}\!\} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \alpha [\![\mathbf{E}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \beta [\![\mathbf{H}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} \Big) \,\mathrm{d}S,$$

$$\ell_{\mathcal{M}}(\mathbf{v}, \mathbf{w}) = \int_{\mathcal{F}_h^0} (\epsilon \mathbf{E}_0 \cdot \mathbf{v} + \mu \mathbf{H}_0 \cdot \mathbf{w}) \,\mathrm{d}S + \int_{\mathcal{F}_h^\partial} (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}) \cdot \big(- \mathbf{w} + \alpha(\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v})\big) \,\mathrm{d}S.$$

Well-posedness and stability identical to wave equation.
Explicit approximation bounds in h. Impedance BCs also fine.
Error bounds in $L^2(Q)^6$ for tent-pitched meshes and impedance.

Symmetric hyperbolic systems

As in MONK–RICHTER: piecewise-constant $\mathsf{A} > 0$, constant A_j

$$\begin{aligned} \mathsf{A}\mathbf{u}_t + \sum_j \mathsf{A}_j \mathbf{u}_{x_j} &= \mathbf{0} && \text{in } \Omega \times (0,T), \\ (\mathsf{D}-\mathsf{N})\mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega \times (0,T), \\ \mathbf{u} &= \mathbf{u}_0 && \text{on } \Omega \times \{0\}, \end{aligned}$$

$\mathsf{D}|_{\partial K} := \sum_j n_K^j \mathsf{A}_j$,

+conditions on N.

Decomposition $\mathsf{M}|_{\partial K} := n_K^t \mathsf{A} + \sum_j n_K^j \mathsf{A}_j = \mathsf{M}_K^+ + \mathsf{M}_K^-$ such that $M^+ \geq 0$, $M^- \leq 0$, $\mathsf{M}_{K_1}^+ + \mathsf{M}_{K_2}^- = 0$ on $\partial K_1 \cap \partial K_2$, leads to

$$\begin{aligned} \mathcal{A}(\mathbf{u},\mathbf{w}) = &\sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} \mathbf{u}_1 \cdot \mathsf{M}_{K_1}^+ (\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}S + \int_{\mathcal{F}_h^T} \mathbf{u} \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S \\ &+ \frac{1}{2} \int_{\partial\Omega \times (0,T)} (\mathsf{D}+\mathsf{N})\mathbf{u} \cdot \mathbf{w}\,\mathrm{d}S, \end{aligned}$$

$$\ell(\mathbf{w}) = -\int_{\mathcal{F}_h^0} \mathbf{u}_0 \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S - \frac{1}{2} \int_{\partial\Omega \times (0,T)} \mathbf{g} \cdot \mathbf{w}\,\mathrm{d}S.$$

$$\begin{aligned} |||\mathbf{u}|||_{DG}^2 := \mathcal{A}(\mathbf{u},\mathbf{u}) = &\sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} (\mathbf{u}_1 - \mathbf{u}_2) \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}S \\ &+ \int_{\mathcal{F}_h^T \cup \mathcal{F}_h^0} \mathbf{u} \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} \mathbf{u}\,\mathrm{d}S + \frac{1}{2} \int_{\partial\Omega \times (0,T)} \mathbf{u} \cdot \mathsf{N}\mathbf{u}\,\mathrm{d}S. \end{aligned}$$

Extensions and open problems

We have described and (a priori) analysed a Trefftz scheme for the wave equation. Basis functions are piecewise-solution polynomials.

- More general space–time meshes (not aligned to t);
- non/less dissipative methods (is our dissipation too much?);
- analysis of non-penalised methods ($\alpha = \beta = 0$);
- L^2 stability in more general cases;
- Maxwell, elasticity, first-order hyperbolic systems, dispersive/Drude-type models for plasmas, . . . ;
- Trefftz hp-approximation theory in dimensions > 1;
- other bases: non-polynomial, trigonometric, directional. . . ;
- (directional) adaptivity;
- . . .

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Thank you!

When does "adjoint stability" hold?

1. 1D, constant c, decomposing solution in left and right waves, $C \sim T(N_x + N_t)^{1/2}$ on a Cartesian-product $N_x \times N_t$ mesh.

2. 1D, general c, with Gronwall + energy + integration by parts +

$$\alpha_{|K_1 \cap K_2} = \frac{\mathtt{a} h^x}{\min\{c^2_{|K_1} h^x_{K_1}, c^2_{|K_2} h^x_{K_2}\}}, \quad \beta_{|K_1 \cap K_2} = \frac{\mathtt{b} h^x}{\min\{h^x_{K_1}, h^x_{K_2}\}}$$

$\Rightarrow C \sim (1/\max_{K \in \mathcal{T}_h}\{h^x_K\} + e^T N^{\text{space}}_{\text{interfaces}})^{1/2}$, hp-type bound.

3. nD, no time-like faces ($\mathcal{F}_h^{\text{time}} = \emptyset$), impedance BCs only,
$\Rightarrow C \sim T\, h_t^{-1/2}$ on uniform meshes.

All bounding constants are explicit.

For general case, need bound on traces of z, $\zeta \cdot \mathbf{n}_x$ in $L^2(\mathcal{F}_h^{\text{time}})$.