Spectrum generated by waveguides in photonic crystals

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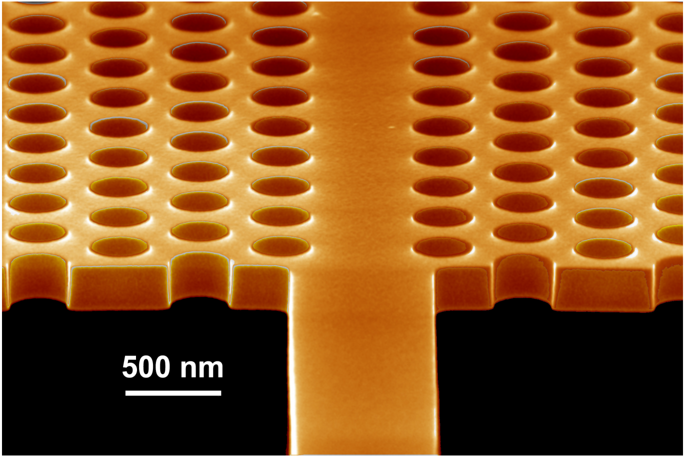
Photonic Crystals

- typically manufactured using periodic crystalline structures
- allow propagation of EM waves only of well-defined frequencies
- band-gap structure of the spectrum

Waveguides

- consider infinite periodic structure with line defect
- line defects can support guided modes which propagate along the defect
- guided modes are confined near defect
- frequencies of guided modes focussed in band gaps



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$$\text{curl } E = -\frac{\partial B}{\partial t}, \quad \text{curl } H = \frac{\partial D}{\partial t}, \quad \text{div } D = 0, \quad \text{div } B = 0.$$

Assumptions:

- $D = \varepsilon E, \quad B = \mu H$, with $\mu \equiv 1$.
- $\varepsilon = \varepsilon(x, y) \geq c > 0$ bounded and independent of z.
- $E(\vec{x}, t) = e^{i\omega t} E(\vec{x})$ and $H(\vec{x}, t) = e^{i\omega t} H(\vec{x})$.

Then

$$\text{curl } E = -i\omega H, \quad \frac{1}{\varepsilon}\text{curl } H = i\omega E, \quad \text{div } (\varepsilon E) = 0, \quad \text{div } H = 0.$$

Next, apply curl :

$$\text{curl curl } E = \omega^2 \varepsilon E, \quad \text{div } (\varepsilon E) = 0$$

$$\mathrm{curl}\ \mathrm{curl}\ E = \omega^2 \varepsilon E, \quad \mathrm{div}\ (\varepsilon E) = 0 \tag{1}$$

Restrict to $\varepsilon = \varepsilon(x, y)$ and to polarized waves $E = (0, 0, u)$. Then

$$\mathrm{curl}\ \mathrm{curl}\ E = (0, 0, -\Delta u), \quad \text{and}$$

$$0 = \mathrm{div}\ (\varepsilon E) = \varepsilon(x, y) \frac{\partial u}{\partial z} \quad \text{implies} \quad u = u(x, y).$$

This reduces (1) to

$$-\Delta u = \omega^2 \varepsilon u \quad \text{or} \quad -\frac{1}{\varepsilon} \Delta u = \omega^2 u \quad \text{on } \mathbb{R}^2.$$

Thus we study the spectral problem for

$$Lu = -\frac{1}{\varepsilon} \Delta u \text{ in } L^2_\varepsilon(\mathbb{R}^2),$$

where

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^2} \varepsilon |u|^2.$$

Consider the spectral problem for the selfadjoint operator L_0 acting on $L^2_{\varepsilon_0}(\mathbb{R}^2)$ given by

$$L_0 u = -\frac{1}{\varepsilon_0(x,y)}\Delta u \quad \text{with} \quad D(L_0) = H^2(\mathbb{R}^2),$$

where $\varepsilon_0(x,y) \geq c > 0$ is bounded and 1-periodic in both x and y. Periodicity in the x-direction allows us to apply the Floquet transform:

$$\mathsf{U}_x : L^2_{\varepsilon_0}(\mathbb{R}^2) \to L^2_{\varepsilon_0}(\Omega \times [-\pi,\pi]),$$

where $\Omega := (0,1) \times \mathbb{R}$, given by

$$(\mathsf{U}_x\, u)(x,y,k_x) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{ik_x n} u(x-n,y)$$

for $x \in [0,1], y \in \mathbb{R}, k_x \in [-\pi,\pi]$. U_x is an isometric isomorphism.

Floquet transform in the x-direction, gives a family of problems:

$$-\frac{1}{\varepsilon_0}\Delta u = \lambda u \quad \text{in } \Omega := (0,1)\times\mathbb{R}$$

with quasiperiodic boundary conditions

$$u(1,y) = e^{ik_x}u(0,y) \quad \text{and} \quad \frac{\partial u}{\partial x}(1,y) = e^{ik_x}\frac{\partial u}{\partial x}(0,y) \qquad (2)$$

for $k_x \in B := [-\pi,\pi]$.
Let $L_0(k_x)$ be the operator acting in $L^2_{\varepsilon_0}(\Omega)$ given by

$$L_0(k_x)u = -\frac{1}{\varepsilon_0(x,y)}\Delta u$$

subject to the quasi-periodic boundary conditions (2). Then

$$L_0 = \int_B^{\oplus} L_0(k_x)\, dk_x \quad \text{and} \quad \sigma(L_0) = \overline{\bigcup_{k_x\in B}\sigma(L_0(k_x))}.$$

For each k_x, due to periodicity in the y-direction, we can take another Floquet transform

$$\mathrm{U}_y : L^2_{\varepsilon_0}(\Omega) \to L^2_{\varepsilon_0}([0,1]^2 \times [-\pi, \pi]),$$

given by

$$(\mathrm{U}_y\, u)(x, y, k_y) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{ik_y n} u(x, y - n)$$

for $x, y \in [0,1], k_y \in [-\pi, \pi]$, giving a family of operators $L_0(k_x, k_y)$ on $L^2_{\varepsilon_0}([0,1]^2)$ subject to qp-bcs in both x and y.
For the spectrum, we have

$$\sigma(L_0(k_x)) = \overline{\bigcup_{k_y \in B} \sigma(L_0(k_x, k_y))} = \overline{\bigcup_n \left(\bigcup_{k_y \in B} \lambda_n(k_x, k_y) \right)}.$$

Thus the spectrum of the operator $L_0(k_x)$ consists of bands.
Any gap in the spectrum of L_0 comes from gaps in the spectra of all $L_0(k_x, k_y)$.

On $L^2_\varepsilon(\mathbb{R}^2)$ consider

- $Lu = -\frac{1}{\varepsilon(x,y)}\Delta u$,
- $\varepsilon(x,y) = \varepsilon_0(x,y) + \varepsilon_1(x,y) > c > 0$ bounded,
- ε_1 supported in $W = \mathbb{R} \times (0,1)$ and 1-periodic in x-direction.

Floquet transform in the x-direction gives family of problems

$$L(k_x)u := -\frac{1}{\varepsilon_0 + \varepsilon_1}\Delta u \tag{3}$$

in $L^2_\varepsilon(\Omega)$ satisfying qp-boundary conditions (2) with $k_x \in B$.
The spectrum of the waveguide problem is given by

$$\sigma(L) = \overline{\bigcup_{k_x \in B} \sigma(L(k_x))}.$$

Aim

Fix k_x and assume (λ_0, λ_1) is a spectral gap for $L_0(k_x)$. Investigate $\sigma(L(k_x)) \cap (\lambda_0, \lambda_1)$.

- Spectral gaps in periodic structures:
  - Existence: Figotin & Kuchment '96, Hoang & Plum & Wieners '09 (Helmholtz), Filonov '03 (Maxwell)
  - Ways of maximizing gap: Cox & Dobson '99 (Helmholtz)
- For compact perturbations:
  - Stability of essential spectrum, creation and estimates on number of gap eigenvalues: Figotin & Klein '96, '98 (Maxwell)
- For line defects:
  - Stability of essential spectrum on the strip, some criteria for existence of eigenvalues: Ammari & Santosa '04 (Helmholtz)
  - Existence of eigenvalues and decay of eigenfunctions away from guide: Kuchment & Ong '04 (Helmholtz), Miao & Ma '07, '08, Kuchment & Ong '10 (Maxwell)

This talk

- Even small perturbations ε_1 lead to eigenvalues being introduced in the gap.
- Only finitely many eigenvalues are introduced, in particular, additional eigenvalues cannot accumulate at the edges of spectral bands.

Consider $L(k_x)u = \lambda u$, i.e.

$$-\Delta u = \lambda(\varepsilon_0 + \varepsilon_1)u \quad \text{on } \Omega = (0,1) \times \mathbb{R}$$

where $\lambda \in (\lambda_0, \lambda_1)$ and all functions satisfy qp-boundary conditions in x.
Equivalently,

$$-\frac{1}{\varepsilon_0}\Delta u - \lambda u = \lambda \frac{\varepsilon_1}{\varepsilon_0} u.$$

λ is an eigenvalue in the gap iff

$$u = \lambda \left(L_0(k_x) - \lambda\right)^{-1} \left(\frac{\varepsilon_1}{\varepsilon_0} u\right) \neq 0.$$

Approach: Study unperturbed strip resolvent $(L_0(k_x) - \lambda)^{-1}$ acting on functions supported in $[0,1]^2$.

Consider

$$L_0(k_x)u = -\frac{1}{\varepsilon_0}\Delta u = \lambda u$$

in $L^2_{\varepsilon_0}(\Omega)$ with qp-boundary conditions in x.
The Floquet transform U_y gives problems on $[0,1]^2$, parametrised by $k \in B$ with qp-bcs in x and y. Let $\{\lambda_s(k)\}_{s\in\mathbb{N}}$ and $\{\psi_s(k)\}_{s\in\mathbb{N}}$ be the eigenvalues and eigenfunctions,
i.e. $L_0(k_x, k)\psi_s(k) = \lambda_s(k)\psi_s(k)$.

Lemma (see Kato)

These are analytic functions in k on B and for each $s \in \mathbb{N}$ they can be continued analytically to a strip in the complex plane

$$\{z \in \mathbb{C} : Re\ z \in (-\pi - \delta, \pi + \delta),\ |Im\ z| < \eta\}$$

containing the interval B.

Proposition

Let $\Sigma = \{(s,k) \in \mathbb{N} \times B : \lambda_s(k) = \lambda_1\}$. Then $|\Sigma|$ is finite.

The Bloch functions are complete: for any $r \in L^2_{\varepsilon_0}(\Omega)$ we have

$$r(\vec{x}) = \frac{1}{\sqrt{2\pi}} \sum_{s\in\mathbb{N}} \int_{-\pi}^{\pi} \langle \mathsf{U}_y\, r(\cdot,k), \psi_s(\cdot,k)\rangle_{\varepsilon_0} \psi_s(\vec{x},k)\; dk.$$

For any $r \in L^2_{\varepsilon_0}((0,1)^2)$ let

$$\begin{aligned} P_s(k,r)(\vec{x}) &:= \langle \mathsf{U}_y\, r(\cdot,k), \psi_s(\cdot,k)\rangle_{\varepsilon_0} \psi_s(\vec{x},k) \\ &= \frac{1}{\sqrt{2\pi}} \langle r(\cdot), \psi_s(\cdot,k)\rangle_{\varepsilon_0} \psi_s(\vec{x},k). \end{aligned}$$

Then

$$(L_0(k_x)-\lambda)^{-1} r = \frac{1}{\sqrt{2\pi}} \sum_{s\in\mathbb{N}} \int_{-\pi}^{\pi} (\lambda_s(k)-\lambda)^{-1} P_s(k,r) dk$$

for λ outside the spectrum of $L_0(k_x)$ (hence for $\lambda \in (\lambda_0,\lambda_1)$) and $r \in L^2_{\varepsilon_0}((0,1)^2)$.

Assumptions:

- $\varepsilon_1 \geq 0$,
- there exists a ball D such that $\inf_D \varepsilon_1 = \alpha > 0$.

Consider

$$u = \lambda \left(L_0(k_x) - \lambda\right)^{-1} \left(\frac{\varepsilon_1}{\varepsilon_0} u\right).$$

Set $v = \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u$. Then v is supported in $[0,1]^2$ and v satisfies

$$v = \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \left(L_0(k_x) - \lambda\right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v.$$

Define A_λ on $L^2_{\varepsilon_0}((0,1)^2)$ by

$$A_\lambda v := \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \left(L_0(k_x) - \lambda\right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v.$$

Aim: Find $\lambda \in (\lambda_0, \lambda_1)$ such that $1 \in \sigma_p(A_\lambda)$.

$$A_\lambda v = \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \left(L_0(k_x) - \lambda\right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v.$$

Lemma

For $\lambda \in (\lambda_0, \lambda_1)$, $A_\lambda : L^2_{\varepsilon_0}((0,1)^2) \to L^2_{\varepsilon_0}((0,1)^2)$ is symmetric and compact.

Set

$$\kappa_{max}(\lambda) = \sup_{\|u\| \neq 0} \frac{\langle A_\lambda u, u \rangle_{\varepsilon_0}}{\langle u, u \rangle_{\varepsilon_0}}.$$

Lemma

Let $\lambda \in (\lambda_0, \lambda_1)$.

1. *$\lambda \mapsto \kappa_{max}(\lambda)$ is continuous.*
2. *$\lambda \mapsto \kappa_{max}(\lambda)$ is monotonically increasing.*

$$\begin{aligned}
\langle A_\lambda u, u\rangle_{\varepsilon_0} &= \lambda \left\langle \varepsilon_0 \left(-\frac{1}{\varepsilon_0}\Delta - \lambda\right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right\rangle_{L^2(\Omega)} \\
&= \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \sum_{s\in\mathbb{N}} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk.
\end{aligned}$$

Now for λ in (λ_0, λ_1), and $s_0 \in \mathbb{N}$ such that λ_1 is the lowest point of the band function $\lambda_{s_0}(\cdot)$,

$$\begin{aligned}
\langle A_\lambda u, u\rangle_{\varepsilon_0} &\leq \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \sum_{s\geq s_0} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk \\
&\leq \frac{\lambda}{2\pi(\lambda_1 - \lambda)} \int_{-\pi}^{\pi} \sum_{s\geq s_0} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk
\end{aligned}$$

$$\begin{aligned}\langle A_\lambda u, u\rangle_{\varepsilon_0} &\leq \frac{\lambda}{2\pi(\lambda_1-\lambda)}\int_{-\pi}^{\pi}\sum_{s\in\mathbb{N}}\left|\left\langle\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}u,\psi_s(\cdot,k)\right\rangle_{\varepsilon_0}\right|^2 dk \\ &\leq \frac{\lambda\,\|\varepsilon_1\|_\infty}{(\lambda_1-\lambda)\inf\varepsilon_0}\,\|u\|_{\varepsilon_0}^2\,.\end{aligned}$$

- If $\|\varepsilon_1\|_\infty \leq \frac{\lambda_1-\lambda_0}{\lambda_0}\inf\varepsilon_0$, then $\kappa_{\max}(\lambda') < 1$ for some $\lambda' \in (\lambda_0, \lambda_1)$.
- Given a fixed λ in the gap, the perturbation needs to have a certain size to make $\kappa_{\max}(\lambda) \geq 1$ (a necessary condition for λ being a gap eigenvalue) and the further λ is from λ_1, the larger this threshold perturbation has to be.

Lower Estimate for $\kappa_{max}(\lambda)$

Let $\lambda_{s_0}(k_0) = \lambda_1 > 0$. There exist $\delta > 0$ and $a > 0$ such that

$$|\langle \psi_{s_0}(\cdot, k_0), \psi_{s_0}(\cdot, k) \rangle_{L^2_{\varepsilon_0}(D)}|^2 \geq a \quad \text{for} \quad k \in (k_0 - \delta, k_0 + \delta).$$

Choose $u = \sqrt{\frac{\varepsilon_0}{\varepsilon_1}} \psi_{s_0}(\cdot, k_0) \chi_D$. Then

$$\begin{aligned}
\frac{\langle A_\lambda u, u \rangle_{\varepsilon_0}}{\|u\|^2_{\varepsilon_0}} &= \frac{\lambda}{2\pi \|u\|^2_{\varepsilon_0}} \int_{-\pi}^{\pi} \sum_{s \in \mathbb{N}} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk \\
&\geq \frac{a\lambda}{2\pi \|u\|^2_{\varepsilon_0}} \int_{k_0 - \delta}^{k_0 + \delta} \frac{dk}{\lambda_{s_0}(k) - \lambda} - C.
\end{aligned}$$

Moreover, with $\lambda_{s_0}(k) \leq \lambda_1 + \alpha_n (k - k_0)^2$

$$\int_{k_0 - \delta}^{k_0 + \delta} \frac{dk}{\lambda_{s_0}(k) - \lambda} \geq \frac{2}{\sqrt{\alpha_n (\lambda_1 - \lambda)}} \arctan\left(\sqrt{\frac{\alpha_n}{\lambda_1 - \lambda}} \delta \right) \to \infty \text{ as } \lambda \nearrow \lambda_1$$

So $\kappa_{\max}(\lambda) \to +\infty$, as $\lambda \to \lambda_1$.

Theorem

Assume that $\varepsilon_1 \geq 0$ and that

$$\|\varepsilon_1\|_\infty < \frac{(\lambda_1 - \lambda_0)\inf \varepsilon_0}{\lambda_0}.$$

Then there exists an eigenvalue of the operator $L(k_x)$ in the spectral gap (λ_0, λ_1) of $L_0(k_x)$.

Proof

Choose ε_1 as above. Then $\kappa_{\max}(\lambda') < 1$ for some λ' in the gap. By the Intermediate Value Theorem, we find $\lambda \in (\lambda', \lambda_1)$ with $\kappa_{\max}(\lambda) = 1$, i.e. λ is an eigenvalue of $L(k_x)$. □

Let $|\Sigma| = |\{(s,k) : \lambda_s(k) = \lambda_1\}| = n.$
Non-degeneracy assumption: $\lambda_s(\tilde{k}) \geq \lambda_1 + \alpha|k - \tilde{k}|^2$ for $(s,k) \in \Sigma,\ \tilde{k}$ close to k

Theorem

Let $\varepsilon_1 \geq 0$ be sufficiently small. Then precisely n eigenvalues are created in the gap.

Outline of proof

- The set $M = \{\psi_s(\cdot,k) : (s,k) \in \Sigma\}$ is linearly independent over D.
- $L = \left\{ u : \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \perp \psi_s(\cdot,k) \quad \text{for all } (s,k) \in \Sigma \right\}$ has codimension n.
- $\langle A_\lambda u, u \rangle_{\varepsilon_0} \leq C \|\varepsilon_1\|_\infty \|u\|^2_{\varepsilon_0}$ for $u \in L$, $\lambda \in (\lambda_0, \lambda_1)$. Hence $C\|\varepsilon_1\|_\infty < 1$ implies $\kappa_{n+1}(\lambda) < 1$.
- $\langle A_\lambda u, u \rangle_{\varepsilon_0} \to \infty$ as $\lambda \nearrow \lambda_1$ for $u \in \text{span}\sqrt{\frac{\varepsilon_0}{\varepsilon_1}}\chi_D M$. Hence $\kappa_n(\lambda) \to \infty$ for $\lambda \to \lambda_1$.

- All results have an analogue for negative perturbations $\varepsilon_1 \leq 0$, where the spectrum appears from the lower end of the gap.
- One would expect the result on generation of spectrum also to hold for large perturbations.

Theorem

For any perturbation ε_1, the eigenvalues of $L(k_x)$ cannot accumulate at the band edges.

- Results carry over to 3D-Helmholtz equation for slab and line defects (with some regularity assumptions on the band functions).
- For TE-modes, the Maxwell equations reduce to divergence form elliptic operators. We have similar results also for this case, making use of Green's operators.

For the wave-guide problem in the plane described by L, the spectrum arises as

$$\sigma(L) = \overline{\bigcup_{k_x \in B} \sigma(L(k_x))}.$$

The eigenvalues depend continuously on the parameter k_x, so

- the band spectrum consists of intervals,
- at most finitely many intervals can be introduced into any gap of the spectrum of the unperturbed problem,
- the spectrum does not contain eigenvalues (Hoang-Radosz '14), so light of these frequencies is transmitted through the structure.

Thank you
for your attention!