Dirac Type Asymptotics for Wavepackets in Periodic Media of Finite Contrast

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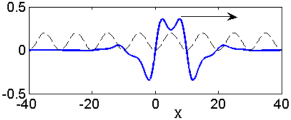
Outline

1. Introduction
   - Wavepackets with one Carrier Bloch Wave $\rightsquigarrow$ NLS asymptotics
   - Wavepackets with two Carrier Bloch Waves $\rightsquigarrow$ Dirac Asymptotics
2. Dirac Asymptotics for Periodic NLS with Finite Contrast
   - Approximation Result
   - Spectral Gap in the Asymptotic Model and the Original Model
   - Numerical Example
3. Future work

Prototype model: periodic Nonlinear Schrödinger equation (PNLS)

$$i\partial_t u + \Delta u - V(x)u - \sigma(x)|u|^2 u = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$
$$V, \sigma \in L^\infty_{\text{per}}((0, 2\pi)^d, \mathbb{R})$$

Overall aim: coherent (solitary) localized pulses propagating across periodic structures
- *not only for infinitesimal contrast!*



(here only $d = 1$)

Linear Waves, Band Structure

- simple example: 1D Schrödinger-Eq.

$$i\partial_t u + \partial_x^2 u - V(x)u = 0, \ x \in \mathbb{R}, \quad V(x+2\pi) = V(x)$$

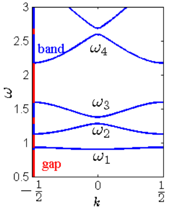
bounded solutions:

Bloch-Waves $\psi_n(x,t;k) = p_n(x,k)e^{i(kx-\omega_n(k)t)}$,

where $p_n(x+2\pi,k) = p_n(x,k), k \in (-1/2, 1/2]$,

$$-(\partial_x + ik)^2 p_n + V(x)p_n = \omega_n(k)p_n$$

Band-Structure



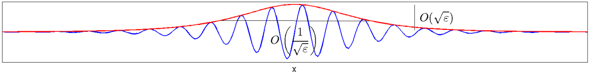
$V(x) = 1 + 0.2(\cos(x) + 3\sin(3x))$

- Bloch waves propagate at the group velocity $v_g(k) = \omega_n'(k)$
- $\omega_n'(k) \neq 0$ if $\omega_n(k) \in \mathrm{int}(\mathrm{spec}(-\partial_x^2 + V))$

Wavepackets with one carrier Bloch wave: NLS asymptotics

$$\mathrm{i}\partial_t u + \partial_x^2 u - V(x)u - \sigma(x)|u|^2 u = 0, \quad x \in \mathbb{R}, t \in \mathbb{R} \qquad \text{(PNLS)}$$

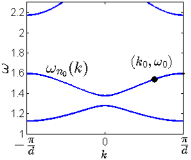
- modulated nearly linear wavepackets = carrier Bloch wave modulated by a small localized slowly varying envelope



<u>formal ansatz:</u> (for $\omega''_{n_0}(k_0) \neq 0$)

$$u(x,t) \sim \sqrt{\varepsilon}A(\sqrt{\varepsilon}(x - v_g t), \varepsilon t)\psi_{n_0}(x,t;k_0) =: u^{app}(x,t) \quad (\varepsilon \to 0)$$

with $v_g = \omega'_{n_0}(k_0)$, $\omega_0 = \omega_{n_0}(k_0)$



<u>effective equation:</u> ($T := \varepsilon t, X := \sqrt{\varepsilon}(x - v_g t)$)

$$\mathrm{i}\partial_T A + \frac{1}{2}\omega''_{n_0}(k_0)\partial_X^2 A + \gamma |A|^2 A = 0, \qquad \text{(NLS)}$$

where $\gamma = -\|\psi_{n_0}(\cdot, k_0)\|^4_{L^4_\sigma(0,d)}$.

NLS asymptotics - rigorous results

Theorem (Pelinovsky, 2011)

Let $V \in C_{per}([0,2\pi],\mathbb{R})$. If $A \in C([0,T],H^6(\mathbb{R}))$ solves NLS, then $\exists \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the solution u of PNLS with $u(x,0) = u^{app}(x,0)$ satisfies $u \in C([0,T/\varepsilon],H^1(\mathbb{R}))$ and

$$\sup_{t\in[0,T/\varepsilon]} \|u(\cdot,t) - u^{app}(\cdot,t)\|_{H^1(\mathbb{R})} \leq C\varepsilon^{3/4}.$$

- periodic nonlinear wave equation
  $\partial_t^2 u = \chi_1(x)\partial_x^2 u - \chi_2(x)u - \chi_3(x)u^3, \quad x \in \mathbb{R}, t \in \mathbb{R}$ (PNLW)

Theorem (Busch, Schneider, Tkeshelashvili, Uecker, 2006)

Let χ_j be smooth and $2\pi-$periodic, $\chi_1,\chi_2 > 0$, and assume the non-resonance condition

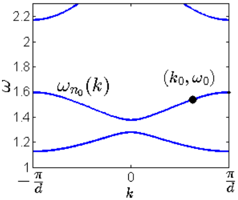
$$|\omega_n(jk_0) - j\omega_{n_0}(k_0)| > \delta > 0 \quad \text{for all } n \in \mathbb{Z}, j \in \{\pm1,\pm3\}, (n,j) \neq \pm(n_0,1).$$

If $A \in C([0,T],H^3(\mathbb{R}))$ solves NLS, then $\exists \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the solution u of PNLW with $u(x,0) = u^{app}(x,0)$ satisfies $u \in C([0,T/\varepsilon],H^1(\mathbb{R}))$ and

$$\sup_{t\in[0,T/\varepsilon]} \|u(\cdot,t) - (u^{app}(\cdot,t) + c.c.)\|_{H^1(\mathbb{R})} \leq C\varepsilon^{3/4}.$$

NLS asymptotics - velocity considerations

- NLS asymptotics of [Pelinovsky 2011, Busch et al, 2006]:



  - Varying k_0 and n_0, one sweeps a range of $v_g = \omega'_{n_0}(k_0)$ but each at a different frequency $\omega_0 = \omega_{n_0}(k_0)$.

- **our aim:**
  - family of wavepackets parametrized by velocity at a fixed frequency

Dirac Asymptotics in 1D PNLS

$$i\partial_t u + \partial_x^2 u - (V(x) + \varepsilon W(x))u - \sigma|u|^2 u = 0, \;\; x \in \mathbb{R} \qquad \text{(1D PNLS)}$$

with $V(x + 2\pi) = V(x)$, $\varepsilon > 0$ and W periodic

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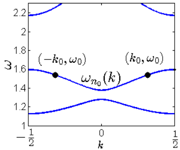
Idea: wavepacket about two counter-propagating Bloch-waves

$$p_{n_0}(x, \pm k_0)e^{\mathrm{i}(\pm k_0 x - \omega_0 t)},$$

which have group velocities $\pm v_g = \omega'_{n_0}(\pm k_0)$



ansatz:

$$u(x,t) \sim \sqrt{\varepsilon}\left[A_+(\varepsilon x, \varepsilon t)p_{n_0}(x, k_0)e^{\mathrm{i}k_0 x} + A_-(\varepsilon x, \varepsilon t)p_{n_0}(x, -k_0)e^{-\mathrm{i}k_0 x}\right] e^{-\mathrm{i}\omega_0 t} =: u^{app}(x,t)$$

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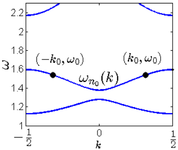

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effective equations: $(X = \varepsilon x, T = \varepsilon t)$

$$\mathrm{i}\left(\partial_T + v_g \partial_X\right) A_+ + \kappa A_- + \alpha(|A_+|^2 + 2|A_-|^2)A_+ + \beta(2|A_+|^2 + |A_-|^2)A_- + \overline{\beta} A_+^2 \overline{A_-} + \gamma A_-^2 \overline{A_+} = 0$$

$$\mathrm{i}\left(\partial_T - v_g \partial_X\right) A_- + \kappa A_+ + \alpha(|A_-|^2 + 2|A_+|^2)A_- + \overline{\beta}(2|A_-|^2 + |A_+|^2)A_+ + \beta A_-^2 \overline{A_+} + \overline{\gamma} A_+^2 \overline{A_-} = 0 \qquad \text{(CME)}$$

$\alpha = -\|p_{n_0}(\cdot, k_0)\|^4_{L^4_\sigma(0,2\pi)}$, $\beta = -\langle \sigma |p_{n_0}(\cdot, k_0)|^2 p_{n_0}(\cdot, -k_0), p_{n_0}(\cdot, k_0)\rangle$,
$\gamma = -\langle \sigma p_{n_0}(\cdot, -k_0)^2 \overline{p_{n_0}(\cdot, k_0)}, p_{n_0}(\cdot, k_0)\rangle$,
$\kappa = -\langle W_1(\cdot) p_{n_0}(\cdot, -k_0), p_{n_0}(\cdot, k_0)\rangle_{L^2(0,d)}$, and $W_1(x) =$(2π-)periodic part of $e^{-2\mathrm{i}k_0 x} W(x)$

Literature on CME

- (CME) have for $v_g, \kappa, \alpha \neq 0, \beta = \gamma = 0$ explicit solitary waves with $v \in (-v_g, v_g)$

[Aceves, Wabnitz, 1989]

$$A_\pm(y,\tau) = f_\pm(y - v\tau)e^{i\theta_\pm(y,\tau)}, \; v \in (-v_g, v_g)$$

with $f_\pm$ and $\theta_\pm$ real, $f_\pm(\xi) \propto \mathrm{sech}(a\xi), a > 0$

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$$\Rightarrow \quad u^{app}(x,t) = \sqrt{\varepsilon}\sum_{\pm} f_{\pm}(\varepsilon(x - vt))e^{\mathrm{i}\theta_{\pm}(\varepsilon x,\varepsilon t)}e^{\mathrm{i}(\pm k_0 x - \omega_0 t)}$$

- range of velocities $v \in (-v_g, v_g)$ *at one frequency* ω_0

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- range of velocities $v \in (-v_g, v_g)$ *at one frequency* ω_0
- rigorous justification of CME for $V = 0, W(x) = \cos(2k_0 x)$
  [Schneider, Uecker, 2001], [Goodman, Weinstein, Holmes, 2001], [Pelinovsky, 2011]

$$\|u(\cdot,t) - u^{app}(\cdot,t)\|_{C_b^0(\mathbb{R})} \leq C\varepsilon^{3/2} \text{ for } t \in [0, T_0/\varepsilon]$$

Coupled mode asymptotics in finite contrast: rigorous result

Theorem (D., Helfmeier 2016)

Let $V, \sigma \in C_{per}([0, 2\pi], \mathbb{R})$, $W(x) = \sum_{m \in \mathbb{N}} a_m e^{im\frac{2\pi}{d}x} + c.c.$, $a_m = 0$ if $m > M$ and let $k_0 \in (0, 1/2]$. If $\hat{A}_\pm \in C^1([0, T_0], L^1_2(\mathbb{R}) \cap L^2(\mathbb{R}))$, and (A_+, A_-) solves CME, then $\exists \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the solution u of PNLS with $u(x, 0) = u^{app}(x, 0)$ satisfies $u \in C([0, T_0/\varepsilon], C^0_b(\mathbb{R}))$, $u(x, t) \to 0$ for $|x| \to \infty$ and

$$\sup_{t \in [0, T_0/\varepsilon]} \|u(\cdot, t) - u^{app}(\cdot, t)\|_{C^0_b(\mathbb{R})} \le C\varepsilon^{3/2}.$$

$$(\|\hat{A}\|_{L^1_q(\mathbb{R})} := \int_{\mathbb{R}} |\hat{A}(k)|(1 + |k|)^q dk)$$

Proof Preparations

Approach: formulation of the equation and ansatz in Bloch variables $\rightsquigarrow$ infinite dimensional ODE system

- Bloch transformation

$$u(x,t) = \sum_{n\in\mathbb{N}} \int_{\mathbb{B}} U_n(k,t) p_n(x,k) e^{\mathrm{i}kx} dk, \qquad U_n(k,t) = \int_{\mathbb{R}^2} u(x,t) \overline{p_n}(x,k) e^{-\mathrm{i}kx} dx$$

  is an isomorphism between $H^s(\mathbb{R}^2)$ and $l^2_s(\mathbb{N}, L^2(\mathbb{B}))$ for $s \geq 0$
- unfortunately L^2-spaces not suitable as $\|f(\varepsilon\cdot)\|_{L^2(\mathbb{R})} = \varepsilon^{-1/2}\|f(\cdot)\|_{L^2(\mathbb{R})}$ (ε-powers lost)
- instead: use L^1 and the fact

$$\vec{U}(\cdot,t) \in \mathcal{X}(s) := l^1_s(\mathbb{N}, L^1(\mathbb{B})), s > 1/2 \;\Rightarrow\; \begin{array}{c} u \in C^0_b(\mathbb{R}),\; |u(x)| < c\|\vec{U}\|_{\mathcal{X}(s)}, \\ u(x) \to 0 \text{ as } |x| \to \infty \end{array}$$

Proof of Theorem (for $W(x) := \cos(2k_0 x)$)

- Bloch mode expansion $u(x,t) = \sum_{n\in\mathbb{N}} \int_{\mathbb{B}} U_n(k,t) p_n(x,k) e^{ikx}\, dx$

PNLS $\Leftrightarrow$

$$\boxed{i\partial_t \vec{U}(k) - \Omega(k)\vec{U}(k) - \varepsilon\left(M^+(k)\vec{U}(k+2k_0) + M^-(k)\vec{U}(k-2k_0)\right) + \vec{N}(\vec{U})(k) = 0,}$$

$\Omega_{i,j}(k) = \delta_{i,j}\omega_j(k), \quad N_j(\vec{U}) = -\langle \sigma(\cdot)(\tilde{u} *_{\mathbb{B}} \bar{\tilde{u}} *_{\mathbb{B}} \tilde{u})(\cdot,k,t), p_j(\cdot,k)\rangle$
$\tilde{u}(x,k,t) = \sum_{n\in\mathbb{N}} U_n(k,t) p_n(x,k)$

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$$\tilde{u}(x,k,t) = \sum_{n\in\mathbb{N}} U_n(k,t) p_n(x,k)$$

- $u^{app}(x,t) = \sqrt{\varepsilon}\Big[A_+(\varepsilon x, \varepsilon t) p_{n_0}(x,k_0) e^{\mathrm{i}k_0 x} + A_-(\varepsilon x, \varepsilon t) p_{n_0}(x,-k_0) e^{-\mathrm{i}k_0 x}\Big] e^{-\mathrm{i}\omega_0 t}$
  - Bloch coefficients of u^{app}:
    $U_n^{app}(k,t) = \varepsilon^{-1/2} \sum_{\pm} \hat{A}_{\pm}(\frac{k\mp k_0}{\varepsilon}, \varepsilon t) \langle p_{n_0}(\cdot, \pm k_0), p_n(\cdot,k)\rangle_{L^2(0,d)}, \quad n \in \mathbb{N}$
- modified/extended ansatz

$$U_{n_0}^{ext}(k,t) := \Big(\varepsilon^{-1/2}\tilde{A}_1(\tfrac{k-k_0}{\varepsilon}, \varepsilon t) + \varepsilon^{1/2}\tilde{A}_{1,3}(\tfrac{k-3k_0}{\varepsilon}, \varepsilon t) + \varepsilon^{-1/2}\tilde{A}_{-1}(\tfrac{k+k_0}{\varepsilon}, \varepsilon t) + \varepsilon^{1/2}\tilde{A}_{-1,-3}(\tfrac{k+3k_0}{\varepsilon}, \varepsilon t)\Big) e^{-\mathrm{i}\omega_0 t}$$

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$\mathrm{supp}(\tilde{A}_j(\cdot, \varepsilon t)) \subset [-\varepsilon^{-1/2}, \varepsilon^{-1/2}]$

$\mathrm{supp}(\tilde{A}_{i,j}(\cdot, \varepsilon t)) \subset [-3\varepsilon^{-1/2}, 3\varepsilon^{-1/2}]$

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- Bloch mode expansion $u(x,t) = \sum_{n\in\mathbb{N}} \int_{\mathbb{B}} U_n(k,t) p_n(x,k) e^{ikx} dx$

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$$\boxed{i\partial_t \vec{U}(k) - \Omega(k)\vec{U}(k) - \varepsilon\left(M^+(k)\vec{U}(k+2k_0) + M^-(k)\vec{U}(k-2k_0)\right) + \vec{N}(\vec{U})(k) = 0,}$$

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$$\vec{U}^{0,ext} := \varepsilon^{-1/2}\left(\tilde{A}_1(\tfrac{k-k_0}{\varepsilon},\varepsilon t) + \tilde{A}_{-1}(\tfrac{k+k_0}{\varepsilon},\varepsilon t)\right) e^{-i\omega_0 t} \mathbf{e}_{n_0}$$

$\mathrm{supp}(\tilde{A}_j(\cdot,\varepsilon t)) \subset [-\varepsilon^{-1/2}, \varepsilon^{-1/2}]$

$\mathrm{supp}(\tilde{A}_{i,j}(\cdot,\varepsilon t)) \subset [-3\varepsilon^{-1/2}, 3\varepsilon^{-1/2}]$

Proof cont.

- residual at $n = n_0, k \in [k_0 - 3\varepsilon^{1/2}, k_0 + 3\varepsilon^{1/2}]$

$$\begin{aligned}\mathrm{Res}_{n_0}(k,t) =& \varepsilon^{1/2}\Big[\Big(\mathrm{i}\partial_T - \varepsilon^{-1}(\omega_{n_0}(k) - \omega_0)\Big)\tilde{A}_1(K,T) + M^+(k)\tilde{A}_{-1}(K,T) \\ &+ \varepsilon^{-1/2}\chi_{[k_0-3\varepsilon^{1/2},k_0+3\varepsilon^{1/2}]}(k)N_{n_0}(\vec{U}^{0,\mathrm{ext}})(k,t)\Big]e^{\mathrm{i}\omega_0 t} + \text{h.o.t.}\end{aligned}$$

- note: $\omega_{n_0}(k) - \omega_0 = \varepsilon\frac{k-k_0}{\varepsilon}v_g(k_0) + O((k-k_0)^2) = \varepsilon K v_g(k_0) + O(\varepsilon^2), \quad K = \frac{k-k_0}{\varepsilon}$

Proof cont.

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  - note: $\omega_{n_0}(k) - \omega_0 = \varepsilon\frac{k-k_0}{\varepsilon}v_g(k_0) + O((k-k_0)^2) = \varepsilon K v_g(k_0) + O(\varepsilon^2), \quad K = \frac{k-k_0}{\varepsilon}$

- Choosing

$$\begin{aligned}\tilde{A}_{\pm 1}(K,T) &:= \chi_{[-\varepsilon^{-1/2},\varepsilon^{-1/2}]}(K)\hat{A}_\pm(K,T)\\ \tilde{A}_{n,j}(K,T) &:= \ldots(\text{explicit functions of } \hat{A}_+(K,T), \hat{A}_-(K,T))\end{aligned}$$

leads to

$$\|\vec{\mathrm{Res}}(\cdot,t)\|_{\mathcal{X}(s)} \leq C_{Res}\varepsilon^{5/2} \text{ for all } t \in [0, T_0\varepsilon^{-1}], s < 1$$

if (A_+, A_-) solves CME and $\hat{A}_\pm \in C^1([0,T_0], L^1_2(\mathbb{R}))$.

Proof cont.

- error: $\vec{R} := \vec{U} - \vec{U}^{ext}$

$$\|\vec{R}(\cdot,t)\|_{\mathcal{X}(s)} \leq c\int_0^t \varepsilon\|\vec{R}(\cdot,t)\|_{\mathcal{X}(s)} + \varepsilon^{1/2}\|\vec{R}(\cdot,t)\|^2_{\mathcal{X}(s)} + \|\vec{R}(\cdot,t)\|^3_{\mathcal{X}(s)} + \varepsilon^{5/2}C_{Res}dt$$

- Gronwall inequality $\Rightarrow$ $\sup_{t\in[0,\varepsilon^{-1}T_0]}\|\vec{R}(\cdot,t)\|_{\mathcal{X}(s)} \leq C\varepsilon^{3/2}$ if $s < 1$
- difference between U^{app} and U^{ext}:
  - for $\hat{A}_\pm(\cdot,T) \in L^1_2(\mathbb{R}) \cap L^2(\mathbb{R})$ is

$$\|U^{app}(\cdot,T) - U^{ext}(\cdot,T)\|_{\mathcal{X}(s)} \leq c\varepsilon^{3/2}$$

□

Do the asymptotics generate a traveling pulse?

- general CME

$$\mathrm{i}\left(\partial_T + v_g\partial_X\right)A_+ + \kappa A_- + \alpha(|A_+|^2 + 2|A_-|^2)A_+ + \beta(2|A_+|^2 + |A_-|^2)A_- + \overline{\beta}A_+^2\overline{A_-} + \gamma A_-^2\overline{A_+} = 0$$
$$\mathrm{i}\left(\partial_T - v_g\partial_X\right)A_- + \overline{\kappa}A_+ + \alpha(|A_-|^2 + 2|A_+|^2)A_- + \overline{\beta}(2|A_-|^2 + |A_+|^2)A_+ + \beta A_-^2\overline{A_+} + \overline{\gamma}A_+^2\overline{A_-} = 0$$

Question:

- Do solitary wave solutions of CME exist?
  - special case: standing waves $(A_+, A_-)(X,T) = e^{-\mathrm{i}\lambda T}(B_+, B_-)(X)$ with $B\pm \in L^2(\mathbb{R})$
    - need $\lambda \in \mathbb{R}$ outside the spectrum of $\begin{pmatrix} \mathrm{i}v_g\partial_X & \kappa \\ \overline{\kappa} & -\mathrm{i}v_g\partial_X \end{pmatrix}$
  - dispersion relation for CME: $\lambda^2(K) = |\kappa|^2 + v_g^2K^2,\ K \in \mathbb{R}$
    $\Rightarrow$ spectral gap $(-|\kappa|, |\kappa|)$ in CME
    $\Rightarrow$ exp-localized profiles $B\pm$ expected (explicitly known for $\beta = \gamma = 0$)

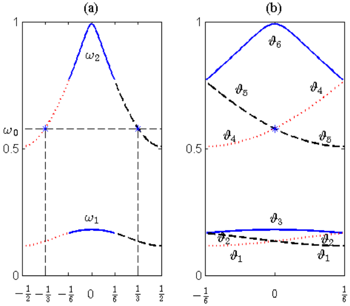
Does a spectral gap in CME imply a spectral gap in PNLS?

1) $k_0 \in \mathbb{Q}$: $V(x)$ and e^{ik_0x} have a common period $P = N2\pi$

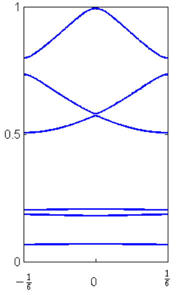
$\rightsquigarrow$ Brillouin zone $\mathbb{B}_P = (-\frac{1}{2N}, \frac{1}{2N}]$

$\Rightarrow \pm k_0 = 0 \mod \frac{1}{N}$, i.e. double eigenvalue at $k = 0$

band str. for $-\partial_x^2 + \cos^5(x)$ ($k_0 = 1/3$ marked)



$\cos^5(x) + \varepsilon\cos(2k_0x), \varepsilon = 0.1$



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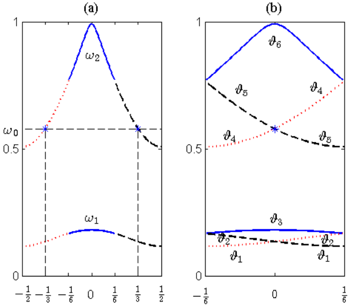

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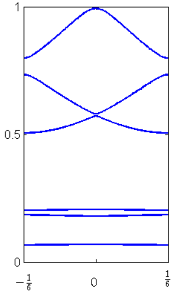
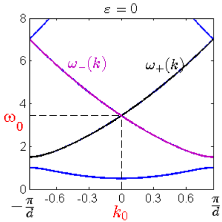


- generically: εW with a period commensurable with P generates a gap in $\sigma(-\partial_x^2 + V + \varepsilon W)$

2) $k_0 \notin \mathbb{Q}$: no common period of $V(x)$ and $e^{ik_0 x}$
$\Rightarrow$ two simple eigenvalues for any period $N2\pi$, $N \in \mathbb{N}$

Does a gap open anyway in $\sigma(-\partial_x^2 + V + \varepsilon W)$?

Construction of a GS-approximation



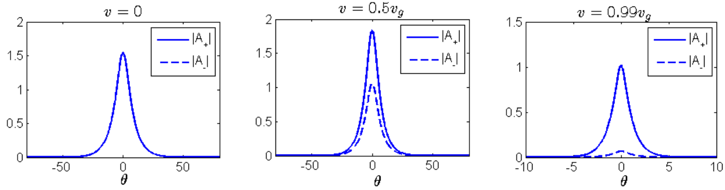
Example: $V(x) = \mathrm{sn}^2(x, 1/2)$ (period $d \approx 3.708$)

$W(x) = \sin(4\pi x/d), k_0 = 0, \omega_0 \approx 3.42$

$$v_g \approx 3.367, \ \kappa \approx 0.493,$$

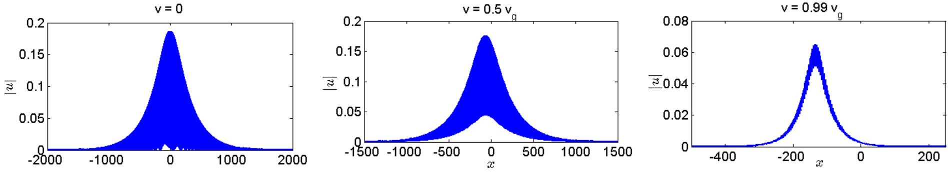
$$\alpha \approx 0.136, \ \beta \approx \mathrm{i}\, 6.48 * 10^{-7}, \ \gamma \approx 7.22 * 10^{-6}$$

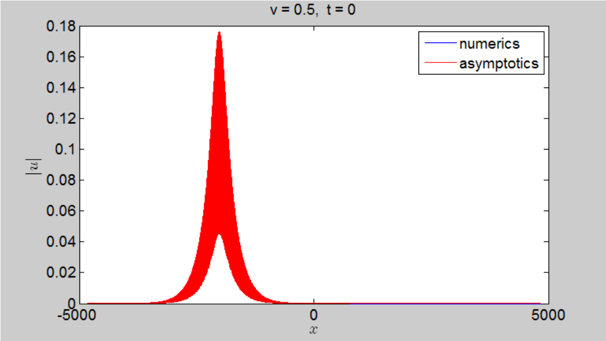
- localized traveling solutions of CME found by numerical homotopy from CME-gap solitons of [Aceves, Wabnitz, 1989] in β, γ in the moving frame variable $\theta = X - v v_g T, \ v \in (-1, 1)$



GS-approximation and the numerical solution [D., SIAM J. Appl. Math, 2014]

$$\varepsilon = 0.025$$





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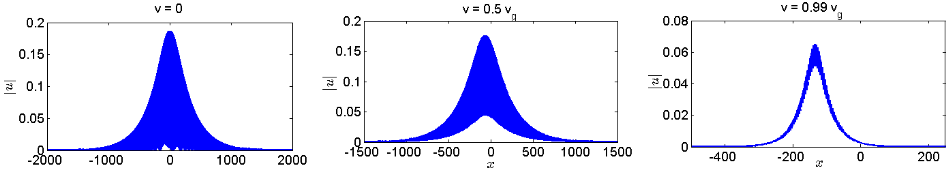

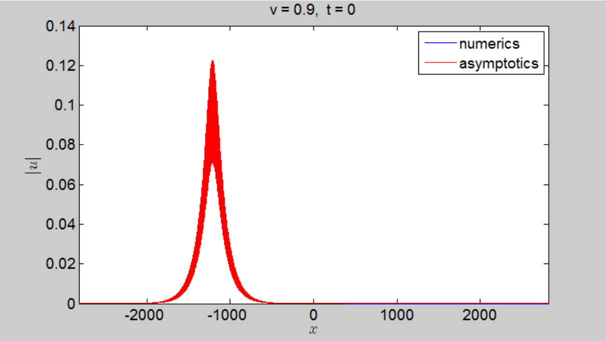

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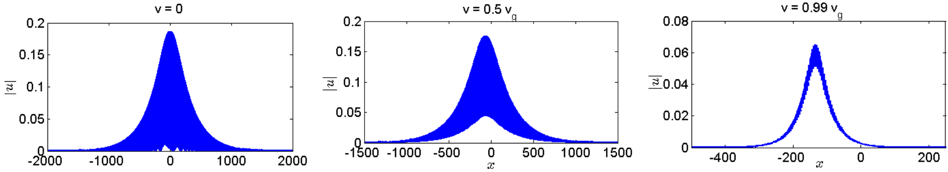

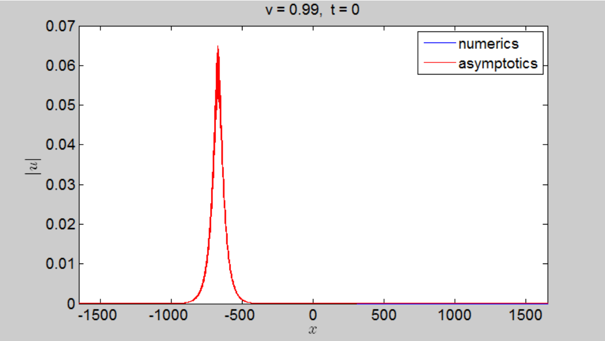

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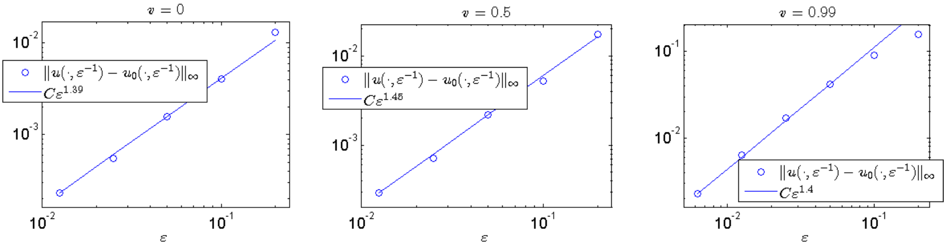


Future work

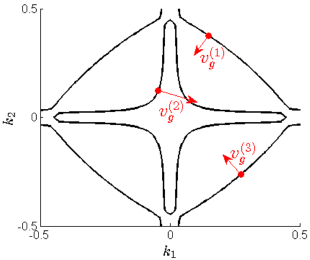
- generalization to higher spatial dimensions:

$$i\partial_t u + \Delta u - (V(x) + \varepsilon W(x)) - \sigma(x)|u|^2 u = 0, x \in \mathbb{R}^n, \quad V, W, \sigma \text{ periodic}$$

Aim: family of pulses traveling at an arbitrary direction in $\mathbb{R}^n$

- Idea: Wavepacket made of $m \geq n+1$ Bloch waves, s.t. $\text{conv}\{v_g^{(1)}, \dots, v_g^{(m)}\}$ contains all directions in $\mathbb{R}^n$

$$u(x,t) \sim \varepsilon^{1/2} \sum_{j=1}^{m} A_j(\varepsilon x, \varepsilon t) \xi_{n_j}(x, k^{(j)})$$



- effective equations

$$i(\partial_\tau A_j + v_g^{(j)} \cdot \nabla_y A_j) + \sum_{n \neq j} \kappa_{j,n} A_n + \mathcal{N}_j(\vec{A}) = 0, \; y \in \mathbb{R}^n, \quad j = 1, \dots, m$$