# Neoclassical Theory of Electromagnetic Interactions II

## One theory for all scales

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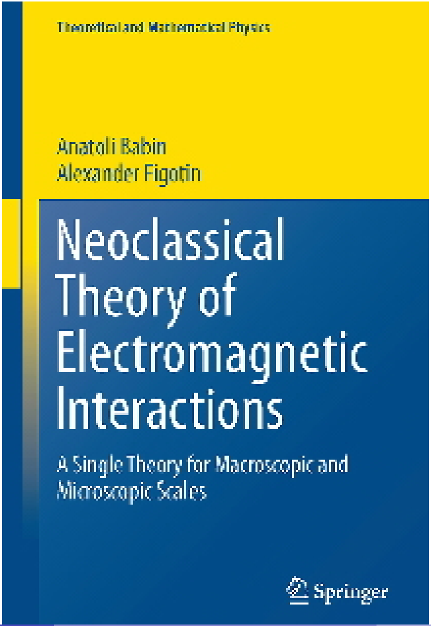
# Our book on neoclassical theory, Springer 2016



# Papers on our neoclassical theory

- B&F, *Wave-Corpuscle Mechanics for Electric Charges*, J. of Stat. Phys., **138**: 912–954, 2010.
- B&F, *Some mathematical problems in a neoclassical theory of electric charges*, Discrete and Continuous Dynamical Systems, **27**, 4, 2010.
- B&F, *Electrodynamics of balanced charges*, Found. of Phys., **41**: 242–260, 2011.
- B&F, *Relativistic dynamics of accelerating particles derived from field equations*, Found. of Phys. **42**:996–1014, 2012.
- B&F, *Relativistic Point Dynamics and Einstein Formula as a Property of Localized Solutions of a Nonlinear Klein–Gordon Equation*, Comm. Math. Phys., **322**: 453–499 (2013).
- B&F, *Newton's law for a trajectory of concentration of solutions to nonlinear Schrödinger equation*, Commun. Pure Appl. Anal. 13, no. 5, 1685–1718 (2014)
- B&F, *Neoclassical theory of elementary charges with spin of 1/2*, J. Math. Phys., **55**: 082901, 1-28 (2014).

# Outline of the presentation

- A concise review of features of the neoclassical theory - balanced charges theory (BCT).
- Lagrangian framework and field equations
- Choosing internal nonlinearity by requiring the Planck-Einstein energy-frequency relation: $E = \hbar\omega$ ($E = \chi\omega$) to be exact in the non-relativistic approximation.
- Derivation on the logarithmic nonlinearity from the Planck-Einstein energy-frequency relation: $E = \hbar\omega$.
- A theorem of multiharmonic solutions for a system of many charges and the Planck-Einstein energy-frequency relation: $E = \hbar\omega$.
- The Schrödinger wave mechanics vs the neoclassical theory.
- Comparison of the neoclassical theory with the QM.
- Uncertainty relations in QM and in the neoclassical theory.

# Neoclassical theory, "balanced charges" theory (BCT)

- The BCT theory is a relativistic Lagrangian theory. It is a single theory for all spatial scales: macroscopic and atomic.
- Balanced charge is a new concept for an elementary charge described by a complex or spinor valued wave function over four dimensional space-time continuum.
- A b-charge does not interact with itself electromagnetically.
- Every b-charge has its own elementary EM potential and the corresponding EM field. It is naturally assigned a conserved elementary 4-current via the Lagrangian.
- B-charges interact with each other only through their elementary EM potentials and fields.
- The field equations for the elementary EM fields are exactly the Maxwell equations with the elementary conserved currents.
- Force densities acting upon b-charges are described exactly by the Lorentz formula.

# New EM features of the neoclassical theory (BCT)

- An elementary charge in the BCT is always a material wave that can acquire particle properties when its energy is localized. This wave function does not have a probabilistic interpretation as in QM, but it can be interpreted as a charge "cloud".
- The coexistence of wave and particle properties in the BCT is manifested through different regimes for the charge wave function. Namely, an elementary charge is in a particle-like state when its energy is well localized, and it is in a wave state when its energy is well spread out in the space.
- Marked difference with the QM duality concept where wave properties are bound to the QM probabilistic aspects.
- The balanced charge is neither a point charge as in classical EM theory and in quantum mechanics, nor is it a distributed charge with a fixed size and geometry as in the Lorentz–Abraham model.

# New EM features of the neoclassical theory (BCT)

- Probabilistic aspects of the theory may arise in it effectively through complex nonlinear dynamical evolution.
- The neoclassical theory has a new fundamental spatial scale - the size of a free electron. Its currently assesed value is 100 Bohr radii - 5 nm.
- Every elementary charge has an individual wave function over four-dimensional space-time continuum, and there is no configuration space as in QM.
- There is no a single EM field as independent entity, instead every elementary charge has its own elementary EM potentials $A^{\ell\mu}$ and corresponding EM fields $F^{\ell\mu\nu}$.
- EM energy is an energy of interaction only defined for any pair of b-charges. The interaction energy density can be positive or negative. The later is analogous to the negative sign of the electrostatic energy for two classical point charges of opposite signs.
- Mechanism of negative radiation for certain prescribed currents.

# More on EM features of the neoclassical theory (BCT)

- The BCT theory accounts for the both coherent (wave) and incoherent (particle) properties of the charged matter.
- The coherent properties are accounted, in particular, in the BCT Hydrogen atom (HA) which has frequency spectrum matching the same for Schrodinger HA with sufficient accuracy.
- Incoherent properties are accounted, in particular, by the Newton equations with Lorentz forces as an approximation in the case when charges are well separated and move with nonrelativistic velocities.
- If a b-charge is in a wave-corpuscle state then the both coherent and incoherent properties are present, namely, there is the de Broglie wave factor manifesting the wave properties and the wave function maintain its spatially localized shape as a particle-like object.

# Lagrangian Field Framework, Particle via Field

Lagrangian density, Variational principles

$$\mathcal{L}(q^{\ell}(x), \partial_{\mu} q^{\ell}(x), x), \ \mu = 0, 1, 2, 3$$

↙ ↘

Euler-Lagrange Field equations

$$\frac{\partial \mathcal{L}}{\partial q^{\ell}} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial q^{\ell}_{,\mu}} = 0$$

Noether theorem: conservation laws via symmetries

$$\partial_{\mu} J_r^{\mu} = 0$$

↓

Energy-momentum (EnM) conservation,
EnM symmetry and the concept of particle (Planck)

$$\partial_{\mu} T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_{\nu}}$$

# Structure of general energy-momentum tensor $T^{\mu\nu}$

- General energy-momentum tensor structure: the energy density $u$, the momentum and the energy flux $p_j$ and $s_j$, the stress tensor $\sigma_{ji}$

$$T^{\mu\nu} = \begin{bmatrix} u & \mathrm{c}p_1 & \mathrm{c}p_2 & \mathrm{c}p_3 \\ \mathrm{c}^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ \mathrm{c}^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \mathrm{c}^{-1}s_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{bmatrix}.$$

- Conservation laws as the motion equations

$$\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} - \frac{\partial \mathcal{L}}{\partial x^i}, \text{ where } p_i = \frac{1}{\mathrm{c}} T^{0i},\ \sigma_{ji} = -T^{ji},\ i,j = 1,2,3,$$

$$\partial_t u + \sum_{j=1,2,3} \partial_j s_j = -\frac{\partial \mathcal{L}}{\partial t}, \text{ where } u = T^{00},\ s_i = \mathrm{c}\,T^{i0} = \mathrm{c}^2 p_i.$$

# Neoclassical theory, electrodynamics of balanced charges

- A system of $N$ elementary b-charges $\left(\psi^{\ell}, A^{\ell\mu}\right)$, $1 \leq \ell \leq N$.
- $\psi^{\ell}$ is the $\ell$-th b-charge wave function (no the configuration space as in the QM!), $A^{\ell\mu}$ and $F^{\ell\mu\nu} = \partial^{\mu} A^{\ell\nu} - \partial^{\nu} A^{\ell\mu}$ are its elementary EM potential and field (no the single EM field as in the CEM!).
- The action upon the $\ell$-th charge by all other charges is described by a single EM potential and field:

$$A_{\neq}^{\ell\mu} = \sum_{\ell' \neq \ell} A^{\ell'\mu}, \quad A_{\neq}^{\ell\mu} = \left(\varphi_{\neq}^{\ell}, \mathbf{A}_{\neq}^{\ell}\right), \quad F_{\neq}^{\ell\mu\nu} = \sum_{\ell' \neq \ell} F^{\ell'\mu\nu},$$

$$F^{\ell\mu\nu} = \partial^{\mu} A^{\ell\nu} - \partial^{\nu} A^{\ell\mu}.$$

- The total EM potential $\mathcal{A}^{\mu}$ and field $\mathcal{F}^{\mu\nu}$:

$$\mathcal{A}^{\mu} = \sum_{1 \leq \ell \leq N} A^{\ell\mu}, \qquad \mathcal{F}^{\mu\nu} = \sum_{1 \leq \ell \leq N} F^{\ell\mu\nu}.$$

- The total EM field is just the sum of elementary ones, it has no independent degrees of freedom which can carry EM energy.

# Lagrangian for many interacting b-charges

- Lagrangian for the system of $N$ b-charges:

$$\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{;\mu}\right\},\left\{\psi^{\ell*},\psi^{\ell*}_{;\mu}\right\},A^{\ell\mu}\right)=\sum_{\ell=1}^{N}L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)+\mathcal{L}_{\mathrm{BCT}},$$

$$\mathcal{L}_{\mathrm{BCT}}=\mathcal{L}_{\mathrm{CEM}}-\mathcal{L}_{\mathrm{e}},\ \mathcal{L}_{\mathrm{CEM}}=-\frac{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}}{16\pi},\ \mathcal{L}_{\mathrm{e}}=-\sum_{1\leq\ell\leq N}\frac{F^{\ell\mu\nu}F^{\ell}_{\mu\nu}}{16\pi},$$

where $L^{\ell}$ is the Lagrangian of the $\ell$-th bare charge, and the covariant derivatives are defined by the following formulas

$$\psi^{\ell}_{;\mu}=\tilde{\partial}^{\ell\mu}\psi^{\ell},\qquad \psi^{\ell*}_{;\mu}=\tilde{\partial}^{\ell\mu*}\psi^{\ell*},$$

$$\tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A^{\ell\mu}_{\neq}}{\chi\mathrm{c}},\qquad \tilde{\partial}^{\ell\mu*}=\partial^{\mu}-\frac{\mathrm{i}q^{\ell}A^{\ell\mu}_{\neq}}{\chi\mathrm{c}}.$$

Covariant differentiation operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{*\mu}$ provide for the "minimal coupling" between the charge and the EM field.

# Lagrangian for many interacting b-charges

- EM part $\mathcal{L}_{\mathrm{BCT}}$ can be obtained by the removal from the classical EM Lagrangian $\mathcal{L}_{\mathrm{CEM}}$ all self-interaction contributions

$$\mathcal{L}_{\mathrm{BCT}} = -\sum_{\{\ell,\ell'\}:\ell'\neq\ell} \frac{F^{\ell\mu\nu}F^{\ell'}_{\mu\nu}}{16\pi} = -\sum_{1\leq\ell\leq N} \frac{F^{\ell\mu\nu}F^{\ell}_{\neq\mu\nu}}{16\pi}.$$

- The "bare" charge Lagrangians (nonlinear Klein-Gordon) $L^{\ell}$ are

$$L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right) = \frac{\chi^2}{2m^{\ell}}\left\{\psi^{\ell*}_{;\mu}\psi^{\ell;\mu} - \kappa^{\ell 2}\psi^{\ell*}\psi^{\ell} - G^{\ell}\left(\psi^{\ell*}\psi^{\ell}\right)\right\},$$

- $G^{\ell}$ is a nonlinear internal-interaction function describing action of internal non-electromagnetic cohesive forces (new physics);
- $m^{\ell} > 0$ is the charge mass; $q^{\ell}$ is the value of the charge;
- $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$ and

$$\kappa^{\ell} = \frac{\omega^{\ell}}{\mathrm{c}} = \frac{m^{\ell}\mathrm{c}}{\chi}, \qquad \omega^{\ell} = \frac{m^{\ell}\mathrm{c}^2}{\chi}.$$

# Euler-Lagrange field equations: elementary wave equations

- Elementary wave equations (nonlinear Klein-Gordon)

$$\left[\tilde{\partial}^{\ell}_{\mu}\tilde{\partial}^{\ell\mu}+\kappa^{\ell 2}+G^{\ell\prime}\left(\left|\psi^{\ell}\right|^{2}\right)\right]\psi^{\ell}=0,\quad \tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A^{\ell\mu}_{\neq}}{\chi \mathrm{c}},$$

and similar equations for the conjugate $\psi^{*\ell}$.

- From the gauge invariance via the Noether theorem we get elementary conserved currents,

$$J^{\ell\nu}=-\mathrm{i}\frac{q^{\ell}}{\chi}\left(\frac{\partial L^{\ell}}{\partial\psi^{\ell}_{;\nu}}\psi^{\ell}-\frac{\partial L^{\ell}}{\partial\psi^{*\ell}_{;\nu}}\psi^{*\ell}\right)=-\mathrm{c}\frac{\partial L^{\ell}}{\partial A^{\ell}_{\neq\nu}},$$

with the conservation laws

$$\partial_{\nu}J^{\ell\nu}=0,\qquad \partial_{t}\rho^{\ell}+\nabla\cdot\mathbf{J}^{\ell}=0,\qquad J^{\ell\nu}=\left(\rho^{\ell}\mathrm{c},\mathbf{J}^{\ell}\right).$$

# Euler-Lagrange field equations: elementary Maxwell equations

- Elementary Maxwell equations

$$\partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{\mathrm{c}} J^{\ell\nu},$$

- or in the familiar vector form

$$\nabla \cdot \mathbf{E}^\ell = 4\pi\varrho^\ell, \qquad \nabla \cdot \mathbf{B}^\ell = 0,$$

$$\nabla \times \mathbf{E}^\ell + \frac{1}{\mathrm{c}}\partial_t \mathbf{B}^\ell = 0, \qquad \nabla \times \mathbf{B}^\ell - \frac{1}{\mathrm{c}}\partial_t \mathbf{E}^\ell = \frac{4\pi}{\mathrm{c}} \mathbf{J}^\ell.$$

- the normalization condition cosistent with the charge conservation in the non-relativistic case takes the form

$$\int \rho_\ell \, \mathrm{d}\mathbf{x} = \mathrm{const} = q_\ell, \qquad \text{or} \qquad \int \psi_\ell \psi_\ell^* \, \mathrm{d}\mathbf{x} = 1,$$

# Euler-Lagrange field equations: charge and current densities

- Elementary currents are just as in QM:

$$J^{\ell\nu} = -\frac{q^{\ell}\chi\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\left(\operatorname{Im}\frac{\partial^{\nu}\psi^{\ell}}{\psi^{\ell}} + \frac{q^{\ell}A_{\neq}^{\ell\nu}}{\chi\mathrm{c}}\right),$$

- or in the vector form

$$\rho^{\ell} = -\frac{q^{\ell}\left|\psi^{\ell}\right|^{2}}{m^{\ell}\mathrm{c}^{2}}\left(\chi\operatorname{Im}\frac{\partial_{t}\psi^{\ell}}{\psi^{\ell}} + q^{\ell}\varphi_{\neq}^{\ell}\right),$$

$$\mathbf{J}^{\ell} = \frac{q^{\ell}\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\left(\chi\operatorname{Im}\frac{\nabla\psi^{\ell}}{\psi^{\ell}} - \frac{q^{\ell}\mathbf{A}_{\neq}^{\ell}}{\mathrm{c}}\right).$$

# Single relativistic charge and the nonlinearity

- Lagrangian (nonlinear Klein-Gordon)

$$L_0 = \frac{\chi^2}{2m}\left\{ \frac{\left|\tilde{\partial}_t \psi\right|^2}{\mathrm{c}^2} - \left|\tilde{\nabla}\psi\right|^2 - \kappa_0^2 \left|\psi\right|^2 - G\left(\psi^*\psi\right) \right\}.$$

- Without EM self-interaction $L_0$ does not depend on the potentials $\varphi$, **A**! Though we can still find the potentials based on the elementary Maxwell equations they have no role to play and carry no energy.
- Rest state of the b-charge

$$\psi\left(t, \mathbf{x}\right) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\mathring{\psi}\left(\mathbf{x}\right), \qquad \omega_0 = \frac{mc^2}{\chi} = \mathrm{c}\kappa_0,$$

$$\varphi\left(t, \mathbf{x}\right) = \mathring{\varphi}\left(\mathbf{x}\right), \qquad \mathbf{A}\left(t, \mathbf{x}\right) = \mathbf{0},$$

  where $\mathring{\psi}\left(\left|\mathbf{x}\right|\right)$ and $\mathring{\varphi} = \mathring{\varphi}\left(\left|\mathbf{x}\right|\right)$ are real-valued radial functions, and we refer to them, respectively, as form factor and form factor potential.
- Rest charge equations:

$$-\nabla^2\mathring{\psi} + G'\left(\left|\mathring{\psi}\right|^2\right)\mathring{\psi} = 0, \qquad -\nabla^2\mathring{\varphi} = 4\pi\left|\mathring{\psi}\right|^2.$$

# Nonlinearity, charge equilibrium equation and its size

- Charge equilibrium equation for the resting charge:

$$-\nabla^2 \mathring{\psi} + G'\left(\left|\mathring{\psi}\right|^2\right)\mathring{\psi} = 0.$$

- It signifies a complete balance of the two forces: (i) internal elastic deformation force $-\Delta\mathring{\psi}$; (ii) internal nonlinear self-interaction $G'\left(|\mathring{\psi}|^2\right)\mathring{\psi}$.
- We pick the form factor $\mathring{\psi}$ considering it as the model parameter and then the nonlinear self interaction function $G$ is determined based on the charge equilibrium equation .
- We integrate the size of the b-charge into the model via size parameter $a > 0$:

$$G_a'(s) = a^{-2} G_1'\left(a^3 s\right), \text{ where } G'(s) = \partial_s G(s).$$

# Choosing internal-interaction nonlinearity

- There is a certain degree of freedom in choosing the form factor and the resulting nonlinearity.
- The proposed choice is justified by its unique physically sound property: the energies and the frequencies of the time-harmonic states of the Hydrogen atom satisfy exactly the Einstein-Planck energy-frequency relation: $E = \hbar\omega$ ($E = \chi\omega$).
- The form factor is Gaussian and defined by

$$\mathring{\psi}(r) = C_g e^{-r^2/2}, \qquad C_g = \frac{1}{\pi^{3/4}},$$

implying

$$\frac{\nabla^2 \mathring{\psi}(r)}{\mathring{\psi}(r)} = r^2 - 3 = -\ln\left(\mathring{\psi}^2(r)/C_g^2\right) - 3.$$

$$\mathring{\varphi}(\mathbf{x}) = q \int_{\mathbb{R}^3} \frac{\left|\mathring{\psi}(\mathbf{y})\right|^2}{|\mathbf{y} - \mathbf{x}|} \, \mathrm{d}\mathbf{y}$$

# Logarithmic internal-interaction nonlinearity and its Gaussian form factor

- Consequently, the nonlinearity reads

$$G'(s) = -\ln\left(s/C_g^2\right) - 3,$$

implying

$$G(s) = -s\ln s + s\left(\ln\frac{1}{\pi^{3/2}} - 2\right).$$

and we call it the logarithmic nonlinearity.

- The nonlinearity explicit dependence on the size parameter $a > 0$ is

$$G_a'(s) = -a^{-2}\ln\left(a^3 s/C_g^2\right) - 3.$$

# Nonrelativistic BCT Lagrangian and its Field Equations

- Non-relativistic Lagrangian

$$\hat{\mathcal{L}}_0\left(\left\{\psi^\ell\right\}_{\ell=1}^N,\left\{\varphi^\ell\right\}_{\ell=1}^N\right)=\frac{\left|\nabla\sum_\ell\varphi^\ell\right|^2}{8\pi}+\sum_\ell\hat{L}_0^\ell\left(\psi^\ell,\psi^{\ell*},\varphi\right),$$

$$\hat{L}_0^\ell=\frac{\chi\mathrm{i}}{2}\left[\psi^{\ell*}\partial_t\psi^\ell-\psi^\ell\partial_t\psi^{\ell*}\right]-\frac{\chi^2}{2m^\ell}\left\{\left|\tilde{\nabla}_{\mathrm{ex}}^\ell\psi^\ell\right|^2+G^\ell\left(\psi^{\ell*}\psi^\ell\right)\right\}-$$

$$-q^\ell\left(\varphi_{\neq\ell}+\varphi_{\mathrm{ex}}\right)\psi^\ell\psi^{\ell*}-\frac{\left|\nabla\varphi^\ell\right|^2}{8\pi},$$

where $\mathbf{A}_{\mathrm{ex}}\left(t,\mathbf{x}\right)$ and $\varphi_{\mathrm{ex}}\left(t,\mathbf{x}\right)$ are potentials of external EM fields.

- The Euler-Lagrange field (non-linear Schrodinger) equations

$$\mathrm{i}\chi\partial_t\psi^\ell=-\frac{\chi^2}{2m^\ell}\left(\tilde{\nabla}_{\mathrm{ex}}^\ell\right)^2\psi^\ell+q^\ell\left(\varphi_{\neq\ell}+\varphi_{\mathrm{ex}}\right)\psi^\ell+\frac{\chi^2}{2m^\ell}\left[G_a^\ell\right]'\left(\left|\psi^\ell\right|^2\right)$$

$$\nabla^2\varphi^\ell=-4\pi q^\ell\left|\psi^\ell\right|^2,\qquad \ell=1,\ldots,N.$$

# Planck-Einstein energy-frequency relation and the nonlinearity

- Let us consider now multiharmonic solutions

$$\psi^{\ell}(t,\mathbf{x}) = \mathrm{e}^{-\mathrm{i}\omega_{\ell} t}\psi_{\ell}(\mathbf{x}), \qquad \varphi^{\ell}(t,\mathbf{x}) = \varphi_{\ell}(\mathbf{x}),$$

$$\frac{1}{4\pi}\nabla^{2}\varphi^{\ell} = -q^{\ell}\left|\psi^{\ell}\right|^{2}, \text{ or } \varphi^{\ell}(t,\mathbf{x}) = q^{\ell}\int_{\mathbb{R}^{3}}\frac{\left|\psi^{\ell}\right|^{2}(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\mathrm{d}\mathbf{y}.$$

- Then $\{\psi_{\ell}(\mathbf{x})\}_{\ell=1}^{N}$ satisfy the nonlinear eigenvalue problem

$$\chi\omega_{\ell}\psi_{\ell} + \frac{\chi^{2}}{2m^{\ell}}\nabla^{2}\psi_{\ell} - q^{\ell}\varphi_{\neq\ell}\psi_{\ell} - \frac{\chi^{2}}{2m^{\ell}}G_{\ell}'\left(|\psi_{\ell}|^{2}\right)\psi_{\ell} = 0.$$

- This problem may have many solutions; every solution $\{\psi_{\ell}\}_{\ell=1}^{N}$ determines a set of frequencies $\{\omega_{\ell}\}_{\ell=1}^{N}$ and energies $\{\mathsf{E}_{0\ell}\}_{\ell=1}^{N}$:

$$\mathsf{E}_{0\ell} = \int q^{\ell}\left|\psi^{\ell}\right|^{2}\varphi_{\neq\ell}\,\mathrm{d}\mathbf{x} + \int\frac{\chi^{2}}{2m^{\ell}}\left\{\left|\nabla\psi^{\ell}\right|^{2} + G^{\ell}\left(\left|\psi^{\ell}\right|^{2}\right)\right\}\mathrm{d}\mathbf{x}.$$

# Planck-Einstein energy-frequency relation and the nonlinearity

- We seek nonlinearities $G^{\ell}$ such that any two solutions $\{\psi_{\ell}\}_{\ell=1}^{N}$, $\{\psi'_{\ell}\}_{\ell=1}^{N}$ satisfy Planck-Einstein frequency-energy relation:

$$\chi\left(\omega_{\ell}-\omega'_{\ell}\right)=\mathrm{E}_{0\ell}-\mathrm{E}'_{0\ell},\qquad \ell=1,\dots,N.$$

- Based on the nonlinear eigenvalue equations and the charge normalization condition $\|\psi_{\ell}\|=1$ we obtain an integral representation for the frequencies $\omega_{\ell}$:

$$\chi\omega_{\ell}=\int\left[\frac{\chi^{2}}{2m^{\ell}}\left|\nabla\psi_{\ell}\right|^{2}\,\mathrm{d}\mathbf{y}+q^{\ell}\varphi_{\neq\ell}\left|\psi_{\ell}\right|^{2}+\frac{\chi^{2}}{2m^{\ell}}G'_{\ell}\left(\left|\psi_{\ell}\right|^{2}\right)\left|\psi_{\ell}\right|^{2}\right]\,\mathrm{d}\mathbf{y}.$$

- Comparing the above with the integral for $\mathrm{E}_{0\ell}$ we see that

$$\chi\omega_{\ell}-\mathrm{E}_{0\ell}=\frac{\chi^{2}}{2m^{\ell}}\int\left[G'_{\ell}\left(\left|\psi_{\ell}\right|^{2}\right)\left|\psi_{\ell}\right|^{2}-G_{\ell}\left(\left|\psi_{\ell}\right|^{2}\right)\right]\,\mathrm{d}\mathbf{y}.$$

# Planck-Einstein energy-frequency relation and the nonlinearity

- Consequently, we obtain for any two solutions $\{\psi_\ell\}_{\ell=1}^N, \{\psi'_\ell\}_{\ell=1}^N$:

$$\chi\left(\omega_\ell - \omega'_\ell\right) - \left(\mathsf{E}_{0\ell} - \mathsf{E}'_{0\ell}\right) =$$
$$= \frac{\chi^2}{2m^\ell} \int G_\ell\left(\left|\psi'_\ell\right|^2\right) - G'_\ell\left(\left|\psi'_\ell\right|^2\right)\left|\psi'_\ell\right|^2 \mathrm{d}\mathbf{y} -$$
$$- \frac{\chi^2}{2m^\ell} \int G_\ell\left(\left|\psi_\ell\right|^2\right) - G'_\ell\left(\left|\psi_\ell\right|^2\right)\left|\psi_\ell\right|^2 \mathrm{d}\mathbf{y}.$$

- Observe that for the above to hold it is sufficient that for every $\left|\psi_\ell\right|^2$ with $\|\psi_\ell\| = 1$, there exists constants $C_\ell$ such that

$$\int \left[G_\ell\left(\left|\psi_\ell\right|^2\right) - G'_\ell\left(\left|\psi_\ell\right|^2\right)\left|\psi_\ell\right|^2\right] \mathrm{d}\mathbf{y} = C_\ell.$$

# Planck-Einstein energy-frequency relation and the nonlinearity

- The previous integral identities will hold if the following differential equations hold for some constants $K_{G_\ell}$:

$$s\frac{d}{ds}G_\ell(s) - G_\ell(s) = K_{G_\ell}s.$$

- The above equation together with the normalization condition $\|\psi_\ell\| = 1$ yield

$$\chi\omega_\ell - \mathrm{E}_{0\ell} = \frac{\chi^2}{2m^\ell}\int\left[G_\ell'\left(|\psi_\ell|^2\right)|\psi_\ell|^2 - G_\ell\left(|\psi_\ell|^2\right)\right]\mathrm{d}\mathbf{y} = -\frac{\chi^2}{2m^\ell}K_{G_\ell},$$

  implying the Planck-Einstein energy-frequency relation.
- Solving the above differential equations we obtain the following explicit formula

$$G_\ell(s) = K_{G_\ell}s\ln s + C_\ell s.$$

  yielding for $K_{G_\ell} < 0$ exactly the logarithmic nonlinearity corresponding to the Gaussian factor.

# Planck-Einstein energy-frequency relation and the nonlinearity

- For the proper choice of constants $K_{G_\ell}$ we obtain

$$G_\ell\left(s\right) = G_{\ell,a}\left(s\right) = -\frac{1}{\left(a^\ell\right)^2} s \ln s + \frac{1}{\left(a^\ell\right)^2} s \left(\ln \frac{1}{\pi^{3/2}} - 2 - 3\ln a\right),$$

  where $a^\ell$ is the size parameter for $\ell$-th charge.
- If $K_G = 0$ then $G_\ell\left(\left|\psi_\ell\right|^2\right)$ is quadratic and the eigenvalue equations turn into the linear Schrödinger equations for which fulfillment of the Planck-Einstein relation is a well-known fundamental property.
- It is remarkable that the logarithmic nonlinearity which is singled out by the fulfillment of the Planck-Einstein relation has a second crucial property: it allows for a Gaussian localized soliton solution.
- The above arguments continue to hold if there is an external time-independent electric field.

# Multiharmonic solutions for a system of many charges

- Field equations without external fields

$$i\chi\partial_t\psi^\ell + \frac{\chi^2}{2m^\ell}\nabla^2\psi^\ell - q^\ell\varphi_{\neq\ell}\psi^\ell = \frac{\chi^2}{2m^\ell}G'_\ell\left(\left|\psi^\ell\right|^2\right)\psi^\ell, \quad \ell = 1, \dots, N,$$

$$\varphi_{\neq\ell} = \sum_{\ell'\neq\ell}\varphi^\ell, \qquad \frac{1}{4\pi}\nabla^2\varphi^\ell = -q^\ell\left|\psi^\ell\right|^2.$$

- The total conserved energy

$$\mathcal{E} = \sum_\ell \mathsf{E}_{0\ell} + \mathcal{E}_{\mathrm{BCT}}$$

$$\mathsf{E}_{0\ell} = \frac{1}{2}\int q^\ell\left|\psi^\ell\right|^2\varphi_{\neq\ell}\,\mathrm{d}\mathbf{x} + \int\frac{\chi^2}{2m^\ell}\left\{\left|\nabla\psi^\ell\right|^2 + G^\ell\left(\left|\psi^\ell\right|^2\right)\right\}\mathrm{d}\mathbf{x}.$$

where $\mathcal{E}_{\mathrm{BCT}}$ is the energy of EM fields, and $\mathsf{E}_{0\ell}$ is $\ell$-th charge energy in the system's field as it was determined before.

# Multiharmonic solutions for a system of many charges

- For a single charge $\mathsf{E}_{0\ell} = \mathcal{E}_\ell = \mathcal{E}$. In the general case $N \geq 2$ the total energy $\mathcal{E}$ does not equal the sum of $\mathsf{E}_{0\ell}$, and the difference between the sum of $\mathsf{E}_{0\ell}$ and the total energy $\mathcal{E}$ coincides with the total energy $\mathcal{E}_{\mathrm{BCT}}$ of EM fields.
- We assume

$$\psi^\ell \in \Xi, \text{ where } \Xi = \left\{ \psi \in H^1\left(\mathbb{R}^3\right) : \|\psi\|^2 = \int |\psi|^2 \, \mathrm{d}\mathbf{x} = 1 \right\}.$$

# Multiharmonic solutions for a system of many charges

- Let us consider now multiharmonic solutions

$$\psi^{\ell}(t,\mathbf{x}) = \mathrm{e}^{-\mathrm{i}\omega_{\ell}t}\psi_{\ell}(\mathbf{x}), \qquad \varphi^{\ell}(t,\mathbf{x}) = \varphi_{\ell}(\mathbf{x}).$$

$$\frac{1}{4\pi}\nabla^2\varphi^{\ell} = -q^{\ell}\left|\psi^{\ell}\right|^2, \text{ or } \varphi^{\ell}(t,\mathbf{x}) = q^{\ell}\int_{\mathbb{R}^3}\frac{\left|\psi^{\ell}\right|^2(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\mathrm{d}\mathbf{y}.$$

- Then $\psi_{\ell}(\mathbf{x})$ satisfy the following nonlinear eigenvalue problem, $\ell = 1, \dots, N$

$$\chi\omega_{\ell}\psi_{\ell} + \frac{\chi^2}{2m^{\ell}}\nabla^2\psi_{\ell} - q^{\ell}\varphi_{\neq\ell}\psi_{\ell} - \frac{\chi^2}{2m^{\ell}}G_{\ell}'\left(|\psi_{\ell}|^2\right)\psi_{\ell} = 0.$$

# Multiharmonic solutions for a system of many charges

## Theorem

*Let* $G_\ell(s)$, $\ell = 1, ..., N$ *be the logarithmic functions with* $a = a^\ell$*. Suppose* $\{\psi_\ell^\sigma\}_{\ell=1}^N \in \Xi^N$, $\sigma \in \Sigma$ *is a set of solutions to the nonlinear eigenvalue problem with the corresponding frequencies* $\{\omega_\ell^\sigma\}_{\ell=1}^N$ *and finite energies* $\{\mathrm{E}_{0\ell}^\sigma\}_{\ell=1}^N$*. Then*

$$\mathrm{E}_{0\ell} = \chi\omega_\ell + \frac{\chi^2}{2\left(a^\ell\right)^2 m^\ell}.$$

*and any two solutions* $\psi_\ell^\sigma = \psi_\ell$ *and* $\psi_\ell^{\sigma_1} = \psi'_\ell$ *with* $\sigma, \sigma_1 \in \Sigma$ *satisfy the Planck-Einstein relation*

$$\chi\left(\omega_\ell^\sigma - \omega_\ell^{\sigma_1}\right) = \mathrm{E}_{0\ell}^\sigma - \mathrm{E}_{0\ell}^{\sigma_1}, \qquad \ell = 1, ..., N.$$

# Schrödinger wave mechanics

- The Schrödinger wave theory was inspired by de Broglie ideas on phase waves. Schrödinger's approach to the construction of wave mechanics is rooted in a deep inner connection of the Hamilton theory and propagation of waves:
- "The inner connection between Hamilton's theory and the process of wave propagation is anything but a new idea. It was not only well known to Hamilton, but it also served him as the starting-point for his theory of mechanics, which grew out of his Optics of Nonhomogeneous Media. Hamilton's variation principle can be shown to correspond to Fermat's Principle for a wave propagation in configuration space (q-space), and the Hamilton–Jacobi equation expresses Huygens' Principle for this wave propagation...".
- He makes then an observation, critical to his entire theory, that geometrical optics by itself is just an approximation to the undulatory (wave) theory, and states that a proper wave equation in the configuration space must replace the fundamental equations of mechanics.

# Schrödinger wave mechanics

- Schrödinger's grand vision of the integration of quantum phenomena into the wave theory was that these phenomena, including line spectra, should arise naturally as "proper" states and "proper" values (resonances, eigenvalues) of a certain wave equation with boundary conditions. Schrödinger wave mechanics is constructed based on the classical point particle Hamiltonian

$$\mathcal{E} = H(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$$

- via the quantization procedure

$$\mathbf{p} \rightarrow -\mathrm{i}\hbar\nabla, \qquad \mathcal{E} \rightarrow \mathrm{i}\hbar\frac{\partial}{\partial t},$$

- yielding the celebrated Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2\psi}{2m} + V(\mathbf{x})\,\psi.$$

# Schrödinger wave mechanics vs the neoclassical theory

- The quantization agrees with Bohr's correspondence principle, which requires a correspondence between quantum and classical mechanics in the limit of large quantum numbers
- Our approach works other way around. We introduce the Lagrangian and the corresponding field equations as a fundamental basis and deduce from them the classical Newtonian mechanics as a certain approximation
- To appreciate the difference consider a system of $N$ charges. In wave-corpuscle mechanics there are $N$ wave functions and the EM fields defined over the same 3 dimensional space, whereas the same system of $N$ charges in the Schrödinger wave mechanics has a single wave function defined over a $3N$-dimensional "configuration space".

# Born's objections to Schrodinger's material wave interpretation of particles

- Born's objections to Schrodinger's material wave interpretation of particles which "lands us in grave difficulties":
- "To begin with, Schrodinger attempted to interpret corpuscles and particularly electrons, as wave packets. Although his formulae are entirely correct, his interpretation cannot be maintained, since on the one hand, as we have already explained above, the wave packets must in course of time become dissipated, and on the other hand the description of the interaction of two electrons as a collision of two wave packets in ordinary three-dimensional space lands us in grave difficulties."
- In BCT there is no configuration space but rather every elementary charge is described by its individual wave function over 4-dimensional space-time continuum.
- In BCT the concept of particle is represented by the concept of a wave-corpuscle which does not disperse. The wave and particle properties naturally coexist in the wave-corpuscle.

# Comparison of the neoclassical theory with the QM

- In sharp contrast to QM, there is no configuration space in our theory. The system of $N$ interacting charges is described by $N$ individual wave functions over the three spatial variables and time. Recall that in QM the same system has one wave function of time and 3N spatial variables known as the configuration space.
- In the neoclassical theory the wave function for every elementary charge does not have a probabilistic interpretation as in QM but rather it can be interpreted as a charge "cloud".
- In particle-like regimes, the balanced charge described by wave-corpuscles is not subjected to dispersion. In contrast, QM wave-function in similar circumstances disperses
- In the neoclassical theory the energy $E$ and the frequency $\omega$ are two independent physical quantities, which for certain regimes are related according to the Planck–Einstein energy-frequency relation $E = \hbar\omega$. In QM, the Planck–Einstein energy-frequency relation $E = \hbar\omega$ is universal and fundamentally exact, and the Schrödinger equation for the entire system is in fact an operator form of the Planck–Einstein energy-frequency relation.

# Comparison of the neoclassical theory with the QM

- The Heisenberg uncertainty principle does not hold as a universal principal in the BCT. Since a balanced charge is not a point, there is an uncertainty in its location, but it is not probabilistic in nature. In the case of the wave-corpuscle, the total momentum of the charge is defined without any uncertainty, whereas the wavevector and position of the charge allow uncertainties which satisfy the uncertainty principle
- Wave-corpuscles in BCT unlike wave-packets in QM do not disperse and when well separated interact as point charges governed by the Newton motion equations with the Lorentz forces. In other words, in relevant regimes classical Newtonian mechanics can be deduced from the field equations as an approximation.

# Comparison of the neoclassical theory with the QM

- In spite of fundamental differences, there is solid common ground between the neoclassical theory and quantum mechanics. This common ground rests on similarities between the neoclassical field equations and the Schrödinger and Klein–Gordon equations in the non-relativistic and relativistic case respectively.
- Common feature of a wavepacket and a wave-corpuscle is the wave mechanism of their motion manifested through the equality of their velocities to the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$ of underlying linear medium with the dispersion relation $\omega(\mathbf{k})$.

# Uncertainty Relation in QM and the neoclassical theory

- There is a marked difference between the essence of the uncertainty relations in quantum mechanics and in the neoclassical theory. For simplicity's sake, let us consider the non-relativistic versions of both theories.
- The difference in the two theories shows already in the origin of the uncertainty.
- In QM the uncertainty is bound to the probabilistic interpretation of the wave function. Namely, an elementary charge is a point-like object, and the position vector $\mathbf{x}$ describes unambiguously its spatial location. The uncertainty of the charge location comes entirely through its complex-valued wave function $\psi(\mathbf{x})$, so that the probability to locate the charge in an infinitesimally small spatial domain of volume $\mathrm{d}\mathbf{x}$ is postulated to be $|\psi(\mathbf{x})|^2 \mathrm{d}\mathbf{x}$.
- Since in QM the wave function describes completely quantum states of a charge, its location is fundamentally uncertain and described in probabilistic terms from the outset: [Holland, p. 72 xvii]: "The uncertainty is postulated to be intrinsic to the system."

# Uncertainty Relation in QM and the neoclassical theory

- In the neoclassical theory an elementary charge is never exactly a point object, but it can be localized and treated then as a point-like object. Its state just as in QM is described by a complex-valued wave function $\psi(\mathbf{x})$, but its physical interpretation is very different from QM one.
- The wave function $\psi(\mathbf{x})$ of the balanced charge is interpreted as a "material" wave of the charged matter distributed in space, and $q|\psi(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}$ *is interpreted as a fraction of the entire charge residing in an infinitesimally small spatial domain of volume* $\mathrm{d}\mathbf{x}$.
- Hence in the neoclassical theory the uncertainty in the charge location originates in its actual spatial distribution over the space. Consequently, the uncertainty here is simply about a natural ambiguity in assigning a single geometric point to a spatially distributed object. Such an uncertainty differs markedly from QM probabilistic uncertainty.