Neoclassical Theory of Electromagnetic Interactions I

One theory for all scales

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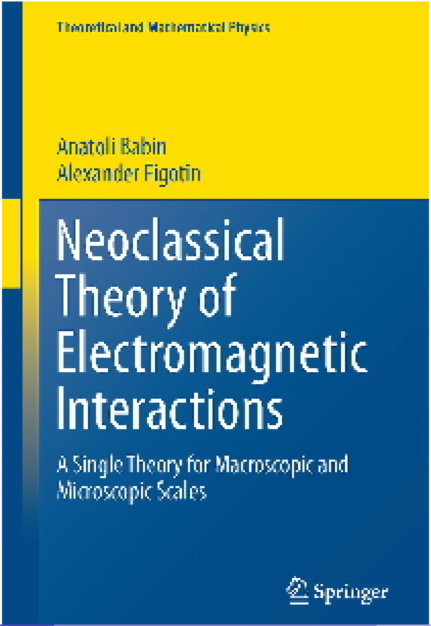

Papers on our neoclassical theory

- B&F, *Wave-Corpuscle Mechanics for Electric Charges*, J. of Stat. Phys., **138**: 912–954, 2010.
- B&F, *Some mathematical problems in a neoclassical theory of electric charges*, Discrete and Continuous Dynamical Systems, **27**, 4, 2010.
- B&F, *Electrodynamics of balanced charges*, Found. of Phys., **41**: 242–260, 2011.
- B&F, *Relativistic dynamics of accelerating particles derived from field equations*, Found. of Phys. **42**:996–1014, 2012.
- B&F, *Relativistic Point Dynamics and Einstein Formula as a Property of Localized Solutions of a Nonlinear Klein–Gordon Equation*, Comm. Math. Phys., **322**: 453–499 (2013).
- B&F, *Newton's law for a trajectory of concentration of solutions to nonlinear Schrödinger equation*, Commun. Pure Appl. Anal. 13, no. 5, 1685–1718 (2014)
- B&F, *Neoclassical theory of elementary charges with spin of 1/2*, J. Math. Phys., **55**: 082901, 1-28 (2014).

Outline of the presentation

- Motivations and justification for a neoclassical theory of EM interactions
- Areas where the neoclassical theory can be useful: plasma physics, plasmonics, emission physics
- Problems with the point charge concept
- Principal features of the neoclassical theory
- Lagrangian framework and field equations
- Nonlinear internal non-electromagnetic interaction associated with every elementary charge
- Wave-corpuscle (soliton, particle-like) exact solution to the field equation
- Size of free electron - new fundamental spatial scale

Why a neoclassical theory of electromagnetic interaction?

- The classical EM theory, based on the Maxwell equations and Lorentz forces for continuously distributed charges and currents holds very well at macroscopic scales. But the classical EM theory treats elementary charges as points and it has a serious divergence problem of the EM field exactly at the positions of the point charges.
- Quantum mechanics (QM) covers EM phenomena at microscopic scales and is confirmed by numerous experiments. But it has a number of issues including "wave function collapse" and that it is inherently non-relativistic.
- Quantum Electrodynamics (QED) has "infinities" which according to P. Dirac "are not to be tolerated".
- To summarize, the EM theory has serious mathematical problems to be addressed.+

Areas where the neoclassical theory can be useful

- Plasma physics. "In its most general sense, a plasma is any state of matter which contains enough free, charged particles for its dynamical behavior to be dominated by electromagnetic forces", Inan U. et. al. "Principles of Plasma Physics"..
- Plasma physics problems: (i) several or many charges moving in the space under EM interaction and collective phenomena in macroscopic and microscopic scales; (ii) ionization, excitation, recombination, and charge exchange at atomic scales; (iii) computer simulations: particle-in-cell (PIC), cloud-in-cell methods.
- Emission physics.
- Plasmonics

Challenges in modeling EM interactions

- EM interactions are significant at all scales: microscopic, macroscopic and astronomical.
- Elementary and statistical aspects of EM interactions are naturally intricately entangled.
- It is very challenging to extract elementary aspect from statistical one experimentally, and such an extraction is always theory dependent. "It is the theory which decides what we can observe", A. Einstein.
- Plasma physics by the very nature of plasma needs an EM theory covering all spatial scales.

Classical Maxwell-Lorentz Electromagnetic Theory (CEM).

- The Maxwell theory described by him in his two-volume treatise had a number of monumental achievments.
- Unification of electric, magnetic and light phenomena.
- Prophetic recognition of the importance of the EM potentials.
- Maxwell being far ahead of his time wrote his equations in quaternion form (Clifford algebra) that represents correctly geometric aspects of his theory.
- Recognition of the importance of the Lagrangian method.

Classical Maxwell-Lorentz EM Theory: Lorentz's contribution

- Lorentz's theory of EM phenomena in a nutshell is a combination of Clausius' theory of electricity with Maxwell's theory of the ether with matter being clearly separated from ether.
- Electrons in his theory do not interact directly but via the electromagnetic field, that is to say, electrons and their motion determine the EM field, and the EM field acts upon the electrons.
- A significant innovation brought by Lorentz was the microscopic treatment of polarization in contrast to Maxwellian physicists treating polarization as a macroscopic property of the ether not extended to the atomic scales.

Classical EM: Point Charge and the EM Field

- Point charge q of the mass m in the EM field

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m\mathbf{v}\left(t\right)\right]=q\left[\mathbf{E}\left(\mathbf{r}\left(t\right),t\right)+\frac{1}{\mathrm{c}}\mathbf{v}\left(t\right)\times\mathbf{B}\left(\mathbf{r}\left(t\right),t\right)\right]$$

  where $\mathbf{r}$ and $\mathbf{v}=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ are respectively charge position and velocity, $\mathbf{E}\left(t,\mathbf{r}\right)$ and $\mathbf{B}\left(t,\mathbf{r}\right)$ are the electric field and the magnetic induction.

- The EM field generated by a moving point charge

$$\frac{1}{\mathrm{c}}\frac{\partial\mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=\mathbf{0},\ \ \nabla\cdot\mathbf{B}=0,$$

$$\frac{1}{\mathrm{c}}\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{B}=-\frac{4\pi}{\mathrm{c}}q\delta\left(\mathbf{x}-\mathbf{r}\left(t\right)\right)\mathbf{v}\left(t\right),$$

$$\nabla\cdot\mathbf{E}=4\pi q\delta\left(\mathbf{x}-\mathbf{r}\left(t\right)\right).$$

Closed Charge-"EM field" system divergence problem

- A closed system "charge-EM field" has a problem: divergence of the EM field exactly at the position of the point charge. Even for the relativistic motion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma m\mathbf{v}(t)\right] = q\left[\mathbf{E}(\mathbf{r}(t), t) + \frac{1}{\mathrm{c}}\mathbf{v}(t) \times \mathbf{B}(\mathbf{r}(t), t)\right],$$

  where $\gamma = 1/\sqrt{1 - \mathbf{v}^2(t)/\mathrm{c}^2}$ is the Lorentz factor, and the system has a Lagrangian the problem still persists.
- This problem is well known and discussed in detail, for instance, in:
  - Durr D. and Teufel S., *Bohmian Mechanics. The Physics and Mathematics of Quantum Theory*, Springer, 2009.
  - Spohn H., *Dynamics of Charged Particles and Their Radiation Field*, Cambridge. 2004.

Lorentz quotation, Section 42 from "Structure of Electron":

"Therefore physicists are absolutely free to form any hypotheses on the properties and size of electrons that may best suit them. You can, for instance, choose the old electron (a small sphere with charge uniformly distributed over the surface) or Parson's ring-shaped electron, endowed with rotation and therefore with a magnetic field; you can also make different hypotheses about the size of the electron. In this connection I may mention that A. H. Compton's experiments on the scattering of γ rays by electrons have led him to ascribe to the electron a size considerably greater than it was formerly supposed to have."

Neoclassical theory, "balanced charges" theory (BCT)

- The BCT theory is a relativistic Lagrangian field theory. It is a single theory for all spatial scales: macroscopic and atomic.
- Balanced charge is a new concept for an elementary charge described by a complex or spinor valued wave function over four dimensional space-time continuum.
- A b-charge does not interact with itself electromagnetically.
- Every b-charge has its own elementary EM potential and the corresponding EM field. It is naturally assigned a conserved elementary 4-current via the Lagrangian.
- B-charges interact with each other only through their elementary EM potentials and fields.
- The field equations for the elementary EM fields are exactly the Maxwell equations with the elementary conserved currents.
- Force densities acting upon b-charges are described exactly by the Lorentz formula.

Neoclassical theory relation to the Classical EM and Quantum theories?

- When charges are well separated and move with nonrelativistic velocities the neoclassical theory can be approximated by dynamics of point charges governed by the Newton equations with the Lorentz forces.
- Radiative phenomena in the neoclassical theory are similar to those in the CEM theory in macroscopic scales.
- The fact that the neoclassical theory captures correctly classical aspects of the EM theory is not that surprising since the Maxwell equations and the Lorentz force expressions are exact in it.
- The Hydrogen Atom spectrum and some other phenomena at atomic scales are described by the neoclassical theory similarly to the Quantum Mechanics (QM).
- The neoclassical theory has a new fundamental spatial scale - the size of a free electron. Its currently assesed value is 100 Bohr radii - 5 nm.

Lagrangian Field Framework, Particle via Field

Lagrangian density, Variational principles

$$\mathcal{L}\left(q^{\ell}\left(x\right), \partial_{\mu} q^{\ell}\left(x\right), x\right), \ \mu = 0, 1, 2, 3$$

$\swarrow$ $\searrow$

Euler-Lagrange Field equations

$$\frac{\partial \mathcal{L}}{\partial q^{\ell}} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial q^{\ell}_{,\mu}} = 0$$

Noether theorem: conservation laws via symmetries

$$\partial_{\mu} J^{\mu}_{r} = 0$$

$\downarrow$

Energy-momentum (EnM) conservation,
EnM symmetry and the concept of particle (Planck)

$$\partial_{\mu} T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_{\nu}}$$

Structure of general energy-momentum tensor

- The structure of general energy-momentum tensor: the u is energy density, p_j and s_j are the momentum and the energy flux respectively, σ_{ji} is the stress tensor

$$T^{\mu\nu} = \begin{bmatrix} u & \mathrm{c}p_1 & \mathrm{c}p_2 & \mathrm{c}p_3 \\ \mathrm{c}^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ \mathrm{c}^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \mathrm{c}^{-1}s_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{bmatrix}.$$

- Conservation laws can be viewed as the motion equations:

$$\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} - \frac{\partial \mathcal{L}}{\partial x^i}, \text{ where } p_i = \frac{1}{\mathrm{c}} T^{0i},\ \sigma_{ji} = -T^{ji},\ i,j = 1,2,3,$$

$$\partial_t u + \sum_{j=1,2,3} \partial_j s_j = -\frac{\partial \mathcal{L}}{\partial t}, \text{ where } u = T^{00},\ s_i = \mathrm{c}T^{i0} = \mathrm{c}^2 p_i.$$

Particle via Field: the symmetry of the energy-momentum tensor

Lanczos C. (The Variational Principles of Mechanics, 4e, 1986): "It was Planck in 1909 who pointed out that the field theoretical interpretation of Einstein's principle can only be the symmetry of the energy-momentum tensor. If the T_{i4} ($i = 1, 2, 3$) (i.e. the momentum density) and the T_{4i}, the energy current, did not agree, then the conservation of mass and energy would follow different laws and the principle $m = E$ could not be maintained.
Nor could a non-symmetric energy-momentum tensor guarantee the law of inertia, according to which the centre of mass of an isolated system moves in a straight line with constant velocity."

Particle via Field: the symmetry of the energy-momentum tensor

Pauli W. (Theory of Relativity, 1958): "This symmetry property leads to a very important result. It follows from $T_{i4} = T_{4i}$, because of (342), that

$$\mathbf{g} = \frac{\mathbf{S}}{c^2}.$$

This is the theorem of the momentum of the energy current, first expressed by Planck[229], according to which a momentum is associated with each energy current. This theorem can be considered as an extended version of the principle of the equivalence of mass and energy. Whereas the principle only refers to the total energy, the theorem has also something to say on the localization of momentum and energy."

Neoclassical theory, electrodynamics of balanced charges

- A system of N elementary b-charges $\left(\psi^{\ell}, A^{\ell\mu}\right)$, $1 \leq \ell \leq N$.
- ψ^{ℓ} is the ℓ-th b-charge wave function (no the configuration space as in the QM!), $A^{\ell\mu}$ and $F^{\ell\mu\nu} = \partial^{\mu} A^{\ell\nu} - \partial^{\nu} A^{\ell\mu}$ are its elementary EM potential and field (no the single EM field as in the CEM!).
- The action upon the ℓ-th charge by all other charges is described by a single EM potential and field:

$$A_{\neq}^{\ell\mu} = \sum_{\ell' \neq \ell} A^{\ell'\mu}, \quad A_{\neq}^{\ell\mu} = \left(\varphi_{\neq}^{\ell}, \mathbf{A}_{\neq}^{\ell}\right),$$

$$F_{\neq}^{\ell\mu\nu} = \sum_{\ell' \neq \ell} F^{\ell'\mu\nu}, \quad \text{where } F^{\ell\mu\nu} = \partial^{\mu} A^{\ell\nu} - \partial^{\nu} A^{\ell\mu}.$$

- The total EM potential $\mathcal{A}^{\mu}$ and field $\mathcal{F}^{\mu\nu}$:

$$\mathcal{A}^{\mu} = \sum_{1 \leq \ell \leq N} A^{\ell\mu}, \qquad \mathcal{F}^{\mu\nu} = \sum_{1 \leq \ell \leq N} F^{\ell\mu\nu}.$$

- The total EM field is just the sum of elementary ones, it has no independent degrees of freedom which can carry EM energy.

Lagrangian for many interacting b-charges

- Lagrangian for the system of N b-charges:

$$\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{;\mu}\right\},\left\{\psi^{\ell*},\psi^{\ell*}_{;\mu}\right\},A^{\ell\mu}\right)=\sum_{\ell=1}^{N}L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)+\mathcal{L}_{\mathrm{BCT}},$$

$$\mathcal{L}_{\mathrm{BCT}}=\mathcal{L}_{\mathrm{CEM}}-\mathcal{L}_{\mathrm{e}},\ \mathcal{L}_{\mathrm{CEM}}=-\frac{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}}{16\pi},\ \mathcal{L}_{\mathrm{e}}=-\sum_{1\leq\ell\leq N}\frac{F^{\ell\mu\nu}F^{\ell}_{\mu\nu}}{16\pi},$$

where L^{ℓ} is the Lagrangian of the ℓ-th bare charge, and the covariant derivatives are defined by the following formulas

$$\psi^{\ell}_{;\mu}=\tilde{\partial}^{\ell\mu}\psi^{\ell},\qquad \psi^{\ell*}_{;\mu}=\tilde{\partial}^{\ell\mu*}\psi^{\ell*},$$

$$\tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A^{\ell\mu}_{\neq}}{\chi\mathrm{c}},\qquad \tilde{\partial}^{\ell\mu*}=\partial^{\mu}-\frac{\mathrm{i}q^{\ell}A^{\ell\mu}_{\neq}}{\chi\mathrm{c}}.$$

Covariant differentiation operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{*\mu}$ provide for the "minimal coupling" between the charge and the EM field.

Lagrangian for many interacting b-charges

- EM part $\mathcal{L}_{\mathrm{BCT}}$ can be obtained by the removal from the classical EM Lagrangian $\mathcal{L}_{\mathrm{CEM}}$ all self-interaction contributions

$$\mathcal{L}_{\mathrm{BCT}} = -\sum_{\{\ell,\ell'\}:\ell'\neq\ell} \frac{F^{\ell\mu\nu}F^{\ell'}_{\mu\nu}}{16\pi} = -\sum_{1\leq\ell\leq N} \frac{F^{\ell\mu\nu}F^{\ell}_{\neq\mu\nu}}{16\pi}.$$

- The "bare" charge Lagrangians L^{ℓ} (nonlinear Klein-Gordon) are

$$L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right) = \frac{\chi^2}{2m^{\ell}}\left\{\psi^{\ell*}_{;\mu}\psi^{\ell;\mu} - \kappa^{\ell 2}\psi^{\ell*}\psi^{\ell} - G^{\ell}\left(\psi^{\ell*}\psi^{\ell}\right)\right\},$$

- G^{ℓ} is a nonlinear internal-interaction function describing action of internal non-electromagnetic cohesive forces (new physics);
- $m^{\ell} > 0$ is the charge mass; q^{ℓ} is the value of the charge;
- $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$ and

$$\kappa^{\ell} = \frac{\omega^{\ell}}{\mathrm{c}} = \frac{m^{\ell}\mathrm{c}}{\chi}, \qquad \omega^{\ell} = \frac{m^{\ell}\mathrm{c}^2}{\chi}.$$

Euler-Lagrange field equations: elementary wave equations

- Elementary wave equations (nonlinear Klein-Gordon)

$$\left[\tilde{\partial}_{\mu}^{\ell}\tilde{\partial}^{\ell\mu}+\kappa^{\ell 2}+G^{\ell\prime}\left(\left|\psi^{\ell}\right|^{2}\right)\right]\psi^{\ell}=0,\quad \tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A_{\neq}^{\ell\mu}}{\chi\mathrm{c}},$$

  and similar equations for the conjugate $\psi^{*\ell}$.

- From the gauge invariance via the Noether theorem we get expressions for elementary conserved currents,

$$J^{\ell\nu}=-\mathrm{i}\frac{q^{\ell}}{\chi}\left(\frac{\partial L^{\ell}}{\partial\psi_{;\nu}^{\ell}}\psi^{\ell}-\frac{\partial L^{\ell}}{\partial\psi_{;\nu}^{*\ell}}\psi^{*\ell}\right)=-\mathrm{c}\frac{\partial L^{\ell}}{\partial A_{\neq\nu}^{\ell}},$$

  with the charge conservation law

$$\partial_{\nu}J^{\ell\nu}=0,\qquad \partial_{t}\rho^{\ell}+\nabla\cdot\mathbf{J}^{\ell}=0,\qquad J^{\ell\nu}=\left(\rho^{\ell}\mathrm{c},\mathbf{J}^{\ell}\right).$$

Euler-Lagrange field equations: elementary Maxwell equations

- Elementary Maxwell equations

$$\partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{\mathrm{c}} J^{\ell\nu},$$

- or in the familiar vector form

$$\nabla \cdot \mathbf{E}^\ell = 4\pi\varrho^\ell, \qquad \nabla \cdot \mathbf{B}^\ell = 0,$$

$$\nabla \times \mathbf{E}^\ell + \frac{1}{\mathrm{c}}\partial_t \mathbf{B}^\ell = 0, \qquad \nabla \times \mathbf{B}^\ell - \frac{1}{\mathrm{c}}\partial_t \mathbf{E}^\ell = \frac{4\pi}{\mathrm{c}} \mathbf{J}^\ell.$$

- the normalization condition cosistent with the charge conservation in the non-relativistic case takes the form

$$\int \rho_\ell \, \mathrm{d}\mathbf{x} = \mathrm{const} = q_\ell, \qquad \text{or} \qquad \int \psi_\ell \psi_\ell^* \, \mathrm{d}\mathbf{x} = 1,$$

Euler-Lagrange field equations: charge and current densities

- Elementary currents expressions are as in QM:

$$J^{\ell\nu} = -\frac{q^{\ell}\chi\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\left(\operatorname{Im}\frac{\partial^{\nu}\psi^{\ell}}{\psi^{\ell}} + \frac{q^{\ell}A_{\neq}^{\ell\nu}}{\chi\mathrm{c}}\right),$$

- or in the vector form

$$\rho^{\ell} = -\frac{q^{\ell}\left|\psi^{\ell}\right|^{2}}{m^{\ell}\mathrm{c}^{2}}\left(\chi\operatorname{Im}\frac{\partial_{t}\psi^{\ell}}{\psi^{\ell}} + q^{\ell}\varphi_{\neq}^{\ell}\right),$$

$$\mathbf{J}^{\ell} = \frac{q^{\ell}\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\left(\chi\operatorname{Im}\frac{\nabla\psi^{\ell}}{\psi^{\ell}} - \frac{q^{\ell}\mathbf{A}_{\neq}^{\ell}}{\mathrm{c}}\right).$$

Single relativistic charge and the nonlinearity

- Lagrangian (nonlinear Klein-Gordon)

$$L_0 = \frac{\chi^2}{2m}\left\{ \frac{\left|\tilde{\partial}_t \psi\right|^2}{\mathrm{c}^2} - \left|\tilde{\nabla}\psi\right|^2 - \kappa_0^2 \left|\psi\right|^2 - G\left(\psi^* \psi\right) \right\}.$$

- Without EM self-interaction L_0 does not depend on the potentials φ, $\mathbf{A}$! Though we can still find the potentials based on the elementary Maxwell equations they have no role to play and carry no energy.
- Rest state of the b-charge

$$\psi(t,\mathbf{x}) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\mathring{\psi}(\mathbf{x}), \qquad \omega_0 = \frac{mc^2}{\chi} = \mathrm{c}\kappa_0,$$

$$\varphi(t,\mathbf{x}) = \mathring{\varphi}(\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) = \mathbf{0},$$

where $\mathring{\psi}(|\mathbf{x}|)$ and $\mathring{\varphi} = \mathring{\varphi}(|\mathbf{x}|)$ are real-valued radial functions, and we refer to them, respectively, as form factor and form factor potential.

- Rest charge equations:

$$-\nabla^2\mathring{\psi} + G'\left(\left|\mathring{\psi}\right|^2\right)\mathring{\psi} = 0, \qquad -\nabla^2\mathring{\varphi} = 4\pi\left|\mathring{\psi}\right|^2.$$

Nonlinearity, charge equilibrium equation and its size

- Charge equilibrium equation for the resting charge:

$$-\nabla^2 \mathring{\psi} + G'\left(\left|\mathring{\psi}\right|^2\right)\mathring{\psi} = 0.$$

- It signifies a complete balance of the two forces: (i) internal elastic deformation force $-\Delta\mathring{\psi}$; (ii) internal nonlinear self-interaction $G'\left(|\mathring{\psi}|^2\right)\mathring{\psi}$.
- We pick the form factor $\mathring{\psi}$ considering it as the model parameter and then the nonlinear self interaction function G is determined based on the charge equilibrium equation .
- We integrate the size of the b-charge into the model via size parameter $a > 0$:

$$G'_a(s) = a^{-2} G'_1\left(a^3 s\right), \text{ where } G'(s) = \partial_s G(s).$$

Choosing self-interaction nonlinearity

- There is a certain freedom in choosing the form factor and the resulting nonlinearity.
- The proposed choice is justified by its unique physically sound property: the energies and the frequencies of the time-harmonic states of the Hydrogen atom should satisfy exactly the Einstein-Planck energy-frequency relation: $E = \hbar\omega$ ($E = \chi\omega$).
- For this choice the form factor is Gaussian and defined by

$$\mathring{\psi}(r) = C_g e^{-r^2/2}, \qquad C_g = \frac{1}{\pi^{3/4}},$$

implying

$$\frac{\nabla^2 \mathring{\psi}(r)}{\mathring{\psi}(r)} = r^2 - 3 = -\ln\left(\mathring{\psi}^2(r)/C_g^2\right) - 3.$$

$$\mathring{\varphi}(\mathbf{x}) = q \int_{\mathbb{R}^3} \frac{\left|\mathring{\psi}(\mathbf{y})\right|^2}{|\mathbf{y} - \mathbf{x}|} \, \mathrm{d}\mathbf{y}$$

Logarithmic internal-interaction nonlinearity and its Gaussian form factor

- Consequently, the nonlinearity reads

$$G'(s) = -\ln\left(s/C_g^2\right) - 3,$$

implying

$$G(s) = -s\ln s + s\left(\ln\frac{1}{\pi^{3/2}} - 2\right).$$

and we call it the logarithmic nonlinearity. It is non-analytic at $s = 0$!.

- The nonlinearity explicit dependence on the size parameter $a > 0$ is

$$G_a'(s) = -a^{-2}\ln\left(a^3 s/C_g^2\right) - 3.$$

Nonrelativistic BCT Lagrangian and its Field Equations

- Non-relativistic Lagrangian

$$\hat{\mathcal{L}}_0\left(\left\{\psi^\ell\right\}_{\ell=1}^N,\left\{\varphi^\ell\right\}_{\ell=1}^N\right)=\frac{\left|\nabla\sum_\ell\varphi^\ell\right|^2}{8\pi}+\sum_\ell\hat{L}_0^\ell\left(\psi^\ell,\psi^{\ell*},\varphi\right),$$

$$\hat{L}_0^\ell=\frac{\chi\mathrm{i}}{2}\left[\psi^{\ell*}\partial_t\psi^\ell-\psi^\ell\partial_t\psi^{\ell*}\right]-\frac{\chi^2}{2m^\ell}\left\{\left|\tilde{\nabla}_{\mathrm{ex}}^\ell\psi^\ell\right|^2+G^\ell\left(\psi^{\ell*}\psi^\ell\right)\right\}-$$

$$-q^\ell\left(\varphi_{\neq\ell}+\varphi_{\mathrm{ex}}\right)\psi^\ell\psi^{\ell*}-\frac{\left|\nabla\varphi^\ell\right|^2}{8\pi},$$

where $\mathbf{A}_{\mathrm{ex}}\left(t,\mathbf{x}\right)$ and $\varphi_{\mathrm{ex}}\left(t,\mathbf{x}\right)$ are potentials of external EM fields.

Nonrelativistic BCT Lagrangian and its Field Equations

- The Euler-Lagrange field equations: non-linear Schrödinger equations for the wave functions ψ^{ℓ} and Poisson equations for the scalar potentials φ^{ℓ}, namely

$$\mathrm{i}\chi\partial_t\psi^{\ell} = -\frac{\chi^2}{2m^{\ell}}\left(\tilde{\nabla}^{\ell}_{\mathrm{ex}}\right)^2\psi^{\ell} + q^{\ell}\left(\varphi_{\neq\ell} + \varphi_{\mathrm{ex}}\right)\psi^{\ell} + \frac{\chi^2}{2m^{\ell}}\left[G^{\ell}_a\right]'\left(\left|\psi^{\ell}\right|^2\right)$$

$$\nabla^2\varphi^{\ell} = -4\pi q^{\ell}\left|\psi^{\ell}\right|^2, \quad \ell = 1, \dots, N.$$

- Consequently, potentials φ^{ℓ} can be represented as

$$\varphi^{\ell}(t,\mathbf{x}) = q^{\ell}\int_{\mathbb{R}^3}\frac{\left|\psi^{\ell}\right|^2(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y}.$$

Exact wave-corpuscle solution for non-relativistic set up

- Let us assume a purely electric external EM field: $\mathbf{A}_{\mathrm{ex}} = 0$, $\mathbf{E}_{\mathrm{ex}}(t, \mathbf{x}) = -\nabla \varphi_{\mathrm{ex}}(t, \mathbf{x})$.
- We define the wave-corpuscle (soliton) ψ, φ by

$$\psi(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}S/\chi}\hat{\psi}, \qquad S = m\mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}) + s_{\mathrm{p}}(t),$$
$$\hat{\psi} = \mathring{\psi}(|\mathbf{x} - \mathbf{r}|), \qquad \varphi = \mathring{\varphi}(|\mathbf{x} - \mathbf{r}|), \qquad \mathbf{r} = \mathbf{r}(t).$$

  where $\mathring{\psi}$ is the Gaussian form factor with the corresponding potential $\mathring{\varphi}$;
- $\mathbf{r}(t)$ is determined from the point charge equation

$$m\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = q\mathbf{E}_{\mathrm{ex}}(t, \mathbf{r}),$$

  and $\mathbf{v}(t)$, $s_{\mathrm{p}}(t)$ are determined by formulas

$$\mathbf{v}(t) = \frac{\mathrm{d}r}{\mathrm{d}t}, \qquad s_{\mathrm{p}}(t) = \int_0^t \left(\frac{m\mathbf{v}^2}{2} - q\varphi_{\mathrm{ex}}(t, \mathbf{r}(t))\right) \mathrm{d}t'.$$

de Broglie factor for accelerating charge

- For purely electric external field we define

$$\mathbf{k}(t) = \int_{\mathbb{R}^3} \operatorname{Im} \frac{\nabla \mathring{\psi}(t, \mathbf{x})}{\mathring{\psi}(t, \mathbf{x})} \left|\mathring{\psi}(t, \mathbf{x})\right|^2 \mathrm{d}\mathbf{x}.$$

- The Fourier transform $\mathcal{F}$ of the wave-corpuscle $\psi(t, \mathbf{x})$:

$$[\mathcal{F}\psi](t, \mathbf{k}) = \exp\left\{\mathrm{i}\mathbf{r}(t)\mathbf{k} - \frac{\mathrm{i}s_{\mathrm{p}}(t)}{\chi}\right\}\left(\mathcal{F}\left[\mathring{\psi}\right]\right)\left(\mathbf{k} - \frac{m\mathbf{v}(t)}{\chi}\right),$$

$$\text{implying } \mathbf{k}(t) = \frac{m\mathbf{v}(t)}{\chi}, \ \mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t).$$

- The charge velocity $\mathbf{v}(t)$ equals the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k}(t))$:

$$\omega(\mathbf{k}) = \frac{\chi\mathbf{k}^2}{2m}, \ \nabla_{\mathbf{k}}\omega(\mathbf{k}) = \frac{\chi\mathbf{k}}{m}, \ \text{implying } \nabla_{\mathbf{k}}\omega(\mathbf{k}(t)) = \mathbf{v}(t).$$

Exact wave-corpuscle (soliton) solution

Theorem

Suppose that $\varphi_{\mathrm{ex}}(t, \mathbf{x})$ *is a continuous function which is linear with respect to* $\mathbf{x}$. *Then the wave-corpuscle wave function and its potential*

$$\psi(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}S/\chi}\hat{\psi}, \qquad S = m\mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}) + s_{\mathrm{p}}(t),$$
$$\hat{\psi} = \mathring{\psi}(|\mathbf{x} - \mathbf{r}|), \qquad \varphi = \mathring{\varphi}(|\mathbf{x} - \mathbf{r}|), \qquad \mathbf{r} = \mathbf{r}(t).$$

is the exact solution to the Euler-Lagrange field equation provided $\mathbf{r}(t)$ *satisfies the point charge equation*

$$m\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = q\mathbf{E}_{\mathrm{ex}}(t, \mathbf{r}),$$

and $\mathbf{v}(t)$, $s_{\mathrm{p}}(t)$ *are determined by formulas*

$$\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}t}, \qquad s_{\mathrm{p}} = \int_0^t \left(\frac{m\mathbf{v}^2}{2} - q\varphi_{\mathrm{ex}}(t, \mathbf{r}(t))\right) \mathrm{d}t'.$$

Newtonian Mechanics of point charges as an approximation

- We introduce the ℓ-th charge position $\mathbf{r}^{\ell}(t)$ and velocity $\mathrm{v}^{\ell}(t)$

$$\mathbf{r}^{\ell}(t) = \mathbf{r}_a^{\ell}(t) = \int_{\mathbb{R}^3} \mathbf{x} \left|\psi_a^{\ell}(t,\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}, \ \mathrm{v}^{\ell}(t) = \frac{1}{q^{\ell}} \int_{\mathbb{R}^3} \mathbf{J}^{\ell}(t,\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

- We find then that the positions and velocities are related exactly as in the point charge mechanics:

$$\frac{\mathrm{d}\mathbf{r}^{\ell}(t)}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathbf{x}\partial_t \left|\psi^{\ell}\right|^2 \mathrm{d}\mathbf{x} = \frac{1}{q^{\ell}} \int_{\mathbb{R}^3} \mathbf{J}^{\ell} \mathrm{d}\mathbf{x} = \mathrm{v}^{\ell}(t),$$

- We obtain the kinematic representation for the total momentum which is exactly the same as for the point charges mechanics

$$\mathrm{P}^{\ell}(t) = \frac{m^{\ell}}{q^{\ell}} \int_{\mathbb{R}^3} \mathbf{J}^{\ell}(t,\mathbf{x})\, \mathrm{d}\mathbf{x} = m^{\ell}\mathrm{v}^{\ell}(t),$$

Newtonian Mechanics of point charges as an approximation

- We obtain the following system of equations of motion for N charges:

$$m^{\ell}\frac{\mathrm{d}^2\mathbf{r}^{\ell}\left(t\right)}{\mathrm{d}^2 t}=q^{\ell}\int_{\mathbb{R}^3}\left[\left(\sum_{\ell'\neq\ell}\mathbf{E}^{\ell'}+\mathbf{E}_{\mathrm{ex}}\right)\left|\psi^{\ell}\right|^2+\frac{1}{\mathrm{c}}\mathbf{v}^{\ell}\times\mathbf{B}_{\mathrm{ex}}\right]\mathrm{d}\mathbf{x}.$$

  The derivation is similar to that of the Ehrenfest Theorem in quantum mechanics,

- Suppose that for every ℓ-th charge the densities $\left|\psi^{\ell}\right|^2$ and $\mathbf{J}^{\ell}$ are localized in a-vicinity of the position $\mathbf{r}^{\ell}\left(t\right)$, and that $\left|\mathbf{r}^{\ell}\left(t\right)-\mathbf{r}^{\ell'}\left(t\right)\right|\geq\gamma>0$ with γ independent on a on time interval $\left[0,T\right]$. Then if $a\longrightarrow 0$ we get

$$\left|\psi^{\ell}\right|^2\left(t,\mathbf{x}\right)\rightarrow\delta\left(\mathbf{x}-\mathbf{r}^{\ell}\left(t\right)\right),\ \mathbf{v}^{\ell}\left(t,x\right)=\mathbf{J}^{\ell}/q^{\ell}\rightarrow\mathrm{v}^{\ell}\left(t\right)\delta\left(\mathbf{x}-\mathbf{r}^{\ell}\left(t\right)\right)$$

Newtonian Mechanics of point charges as an approximation

- We infer then

$$\varphi^{\ell}(t,\mathbf{x}) \to \varphi_0^{\ell}(t,\mathbf{x}) = \frac{q^{\ell}}{\left|\mathbf{x}-\mathbf{r}^{\ell}\right|},\ \nabla_{\mathbf{r}}\varphi^{\ell}(t,\mathbf{x}) \to \frac{q^{\ell}\left(\mathbf{x}-\mathbf{r}^{\ell}\right)}{\left|\mathbf{x}-\mathbf{r}^{\ell}\right|^{3}} \text{ as } a \to 0.$$

- Passing to the limit $a \to 0$ we get Newton's equations of motion for point charges

$$m^{\ell}\frac{\mathrm{d}^2\mathbf{r}^{\ell}}{\mathrm{d}t^2} = \mathrm{f}^{\ell},$$

where f^{ℓ} are the Lorentz forces

$$\mathrm{f}^{\ell} = \sum_{\ell'\neq\ell} q^{\ell}\mathbf{E}_0^{\ell'}\left(\mathbf{r}^{\ell}\right) + q^{\ell}\mathbf{E}_{\mathrm{ex}}\left(\mathbf{r}^{\ell}\right) + \frac{1}{\mathrm{c}}\mathrm{v}^{\ell}\times\mathbf{B}_{\mathrm{ex}}\left(\mathbf{r}^{\ell}\right),\ \ell = 1, \dots, N,$$

Full relativistic version of the theory

- Expression for the rest mass m_0 of a charge differs slightly from the mass parameter m

$$m_0 = m + \frac{m}{2}\frac{a_C^2}{a^2}, \qquad a_C = \frac{\chi}{mc},$$

where a is the size of the free charge and a_C is the reduced Compton wavelength assuming that $\chi = \hbar$.

- Electron has a sequence of rest states with only one of minimal energy being stable. The Einstein-Plank relation becomes approximate

$$\mathrm{E}_{0\omega} = \chi\omega\left(1 + \Theta\left(\omega\right)\right), \qquad \Theta\left(\omega\right) = \frac{a_C^2}{2a^2}\frac{\omega_0^2}{\omega^2}, \qquad a_C = \frac{\chi}{mc},$$

$$\omega_0 = \frac{mc^2}{\chi}, \qquad a_C = \frac{\chi}{mc}.$$

Full relativistic version of the theory

- The QM and QED allow to interpret the spectroscopic data yielding the recommended value of the electron mass

$$m = m_{\mathrm{e}} = A_{\mathrm{r}}(e) = 5.4857990943\,(23) \times 10^{-4} \; \left[4.2 \times 10^{-10}\right]$$

- Penning trap measurements can be considered as the most direct measurement of the electron mass as the inertial one

$$A_{\mathrm{r}}(e) = 5.485799111\,(12) \times 10^{-4} \; \left[2.1 \times 10^{-9}\right].$$

- From standard statistical analysis we obtain

$$m_0 - m = \delta A_{\mathrm{r}}(e) = 0.17(12) \times 10^{-11}, \; \left[3.1 \times 10^{-9}\right].$$

implying the size a of a free electron is of order $5\ \mathrm{nm}$ $(100 a_{\mathrm{B}})$.

Electron size and plasma significant spatial scales

- Debye length λ_{D} for shielding effects.
- Collision related parameters: mean-free pass λ_{mph}, average distance between particles r_{d} and the distance of closest approach r_{c}.
- Our concept of electron as "jelly-like" continuously distributed entity has similarities with a conventional plasmonic excitation as well as the concept of "cloud" in the plasma physics.
- "The new fundamental scale in our theory - the size a of a free electron - is significant for collisions, and, just as a "cloud" in plasma, it has very different collision signature compare to that of the point charge. Consequently, the free electron size a (estimated currently as $5\mathrm{nm}$) may play an important role in plasma physics.

Experimental evidence of extremely large currents for nanometer-scale emitting areas

- Fursey (pp. X-XI). "Current densities up to $5 \times 10^9\ \mathrm{A/cm^2}$ were demonstrated for field emission localized to nanometer-scale emitting areas. In experiments by V. N. Shrednik et al., current densities up to $\left(10^9 - 10^{10}\right)\ \mathrm{A/cm^2}$ are recorded from nanometer-sized tips under steady-state conditions. Recently, G. N. Fursey and D. V. Galazanov, using tips with an apex radius of $\sim 10\ \dot{A}$, were able to reach current densities of $\left(10^{10} - 10^{11}\right)\ \mathrm{A/cm^2}$. These current densities are close to the theoretical supply limit of a metal's conduction band when the electron tunneling probability is unity."
- So, it conceivable that micro-machining at the scale comparable with the size of our free electron ($5\mathrm{nm}$) may significantly reduce the work function of the material surface.

"Cloud", a concept of a finite-size charge in plasma physics

- Vlasov A. A., *Many-Particle Theory and Its Application to Plasmas*, Gordon, 1961
- Birdsall C. Fuss D., *Clouds-in-clouds, clouds-in-cells physics for many-body plasma simulation*, JCP, 1969.
- Okuda H. Birdsall C., *Collisions in a Plasma of Finite-Size Particles*, PF, 1970.
- Okuda H., *Verification of Theory of Plasma of Finite-Size Particles*, PF, 1972.
- Dawson J., *Particle simulation of plasmas*, RMP, 1983.