Scattering amplitudes in Lifshitz spacetime

Yang Lei

Durham University

Nov 15th, 2014

Based on 1406.6389, together with Simon F. Ross and Tomas Andrade

Introduction

- Motivation
- Scattering amplitudes in AdS/CFT
- Lifshitz spacetime

Motivation

Lifshitz spacetime in Poincare coordinate

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2 + dx_i^2}{r^2}$$

has non-relativistic scaling relation ($z > 1$);

$$r \to \lambda r; \quad x_i \to \lambda x_i; \quad t \to \lambda^z t$$

Boundary at $r = 0$.

Singularity

Lifshitz spacetime has null singularity at Poincare horizon $r = \infty$. This forbids the geometric extension which goes through this surface.

Tidal force singularity

The singularity at Poincare horizon has divergent tidal force while all the curvature scalars are finite. Understanding this singularity is related to black hole information resolution. Hawking evaporation is not unitary based on the assumption observer falls into black hole without feeling anything special.

Question

Any physical objects are sensible to this IR singularity? What does this singularity mean in the dual field theory? We need to find a physical observable to probe this singularity.

Amplitudes

Amplitudes are promising objects. Intrinsically Lorentian observable.

Scattering amplitudes in flat spacetime at high energy

Classically, string scattering amplitudes are approximated by stationary point

$$\mathcal{A} \sim e^{-S_c}$$

where S_c is found by solving classical equation of motion of the effective action

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \partial_a X_\mu \partial^a X^\mu + i \sum_{i=1}^{n} k_i^\mu X_\mu(\sigma_i)$$

One would find

$$\mathcal{A} \sim \exp[-\alpha'(s \ln s + t \ln t + u \ln u)]$$

Scattering amplitudes in AdS/CFT

Maldacena and Alday (0705.0303) considered massless *open* string scattering in AdS spacetime as the dual to gluon scatting amplitudes in $\mathcal{N} = 4$ SYM theories at strong couplings.
The difficulty is the stationary point equation is too difficult to solve!

Tool

Use AdS/CFT: What questions should be asked?

T-dual

By performing T-dual, the effective action evaluated at the stationary

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \, G_{\mu\nu} \partial_a x^\nu \partial^a x^\mu + i \sum_{i=1}^{n} k_i^\mu x_\mu(\sigma_i)$$

is equivalent to finding the minimal surface at the dual geometric spacetime $\{y^\mu\}$. The boundary of the minimal surface is given by momentum k_i^μ

$$\partial_\alpha y^\mu = i\omega^2(z)\epsilon_{\alpha\beta}\partial^\beta x^\mu$$

where $ds^2 = \omega^2(z) dx^i dx_i + ...$

AdS

In the specific case of AdS geometry, one can find geometric spacetime spanned by $\{y^\mu\}$ is again AdS. And $\Delta y^\mu = 2\pi k^\mu$ works as the boundary condition. They form a polygon with cusps. Our minimal surface in AdS must end on this polygon at the boundary. The polygon is closed because of momentum conservation. Massless gluon requires $k^2 = 0$ which means the segments of this polygon are lightlike.

AdS_3 example

Consider light-like segments near a single cusp $t = \pm x$. We want to consider minimal surface ending on this cusp at the boundary. To parametrize the string worldsheet, we pick

$$t = e^{\tau} \cosh \sigma, \qquad x = e^{\tau} \sinh \sigma, \qquad r = e^{\tau} w(\tau)$$

This ansatz makes full use of the symmetry of this system. The boost symmetry is a translation in σ and scaling symmetry is translation in τ.

Solution

$$r^2 = 2(t^2 - x^2) \qquad \text{(hep-th/0210115)}$$

I will write this solution in the following scaling ansatz.

$$u = \sqrt{1 - \frac{f^2}{2}} \quad \text{with } x = ut, \quad r = ft$$

Evaluate action in AdS_3

Scattering amplitudes are evaluated by the solution above.

$$\mathcal{A} \sim e^{iS}$$

$$iS = -S_E = \frac{R^2}{4\pi} \int d\sigma d\tau \sim \sqrt{\lambda} \int \frac{dx^+ dx^-}{x^+ x^-}$$

Therefore, the light-like segments would generate double log divergence in the scattering amplitudes $\ln^2 \epsilon$.

4-gluon scattering case

Consider the Wilson loop with 4 light-like cusps at the boundary. Consider AdS_5 in Poincare coodinates (r, y_0, y_i). The string worldsheet is parametrized by y_1, y_2. The Nambu-Goto action is

$$S = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2} \tag{1}$$

with boundary condition

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1$$

Solutions

The solution to N-G action above is

$$r^2 = (1 - y_1^2)(1 - y_2^2); \qquad y_0 = y_1 y_2 \tag{2}$$

AdS with four cusps

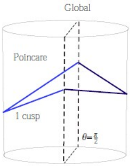

Figure 1:

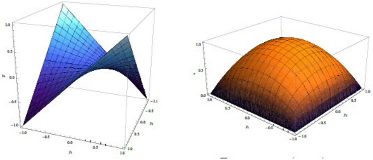

Figure 2:

Before start

We need to resolve two problems before performing the calculation.

- What do we mean by cusp in Lifshitz? In AdS, the light-like cusp is determined by dispersion relation of massless gluons $\omega^2 = k^2$. In Lifshitz field theory, the dispersion relation by Lifshitz symmetry turns out to be $\omega^2 \sim k^{2z}$. Therefore, cusp in Lifshitz should be $t \sim \pm x^z$.
- In AdS, people find minimal surface ending on 4 cusps polygon. Naively, this is hardly to be the case in Lifshitz spacetime. This is because Poincare coordinate in AdS only permit one cusp. 4-cusps are possible because AdS can be extended to global coordinates. The singularity at $r = \infty$ forbids this extension in Lifshitz spacetime.

Ansatz and action

Solutions in AdS spacetime are much easier to find due to larger groups of symmetries. Lifshitz spacetime does not have boost symmetry. Therefore, to parametrize the string world sheet by two coordinates, scaling ansatz would be the best choice we have

$$t = \sigma^z; \quad x = \sigma u(f); \quad r = \sigma f \qquad (t \geq 0)$$

Boundary is at $r = f = 0$. f and u are scaling independent variables. We are going to parametrize string world sheet by σ and f. The boundary conditions are

$$u(0) = u_0; \qquad u(f_0) = 0; \qquad u'(f_0) = \infty \tag{3}$$

u_0 is the proportional constant in the dispersion relation of Lifshitz field theory. Equivalent to

$$f(\pm u_0) = 0 \tag{4}$$

Ansatz and action

The action

$$
\begin{aligned}
S &= \frac{1}{2\pi\alpha'} \int dX_a dX_b \sqrt{\det(G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)} && (5) \\
&= \frac{1}{2\pi\alpha'} \int \frac{d\sigma}{\sigma} \int \frac{df}{f^{z+1}} \,. && (6) \\
&\quad \sqrt{(z^2 - f^{2z})u'^2 + 2uf^{2z-1}u' + (z^2 - u^2 f^{2z-2})}
\end{aligned}
$$

This action requires evaluation by stationary point solutions.

Equation

Stationary point equation is

$$
\begin{aligned}
& f[f^2(f^{2z}-z^2)+f^{2z}u^2]u''+f^2(z+1)(z^2-f^{2z})u'^3 \\
+ \quad & f^{2z+1}(3z+1)uu'^2+[f^2(f^{2z}(z-1)+z^2(z+1))-2zf^{2z}u^2]u' \\
- \quad & (z-1)f^{2z+1}u=0.
\end{aligned}
$$

There is a singularity $f^2(f^{2z}-z^2)+f^{2z}u^2=0$

AdS with timelike cusp

Note in Lifshitz spacetime, the boundary cusp is timelike. We'd better to analyze AdS timelike boundary conditions ($u_0 < 1$) to do the comparison (Remeber $x = u_0 t^{1/z}$ at the boundary). In $z = 1$ case, solve the equation by series expansion

$$u(f) = u_0 + u_3 f^3 + \sum_{i=5} u_i f^i, \tag{7}$$

where u_0 and u_3 are free data and the first few subleading terms are

$$u_5 = -\frac{3u_3}{5(u_0^2 - 1)}, \qquad u_6 = -\frac{2u_3^2 u_0}{(u_0^2 - 1)}. \tag{8}$$

We find solution exists in the range $f_0 < 1$.

AdS with timelike cusp

If we pick cutoff $u = u_0 - \frac{\epsilon}{\sigma}$,

$$S \sim \frac{\sqrt{1 - u_0^2}}{\epsilon^{1/3}}$$

stronger than light-like case (which was double log divergence).

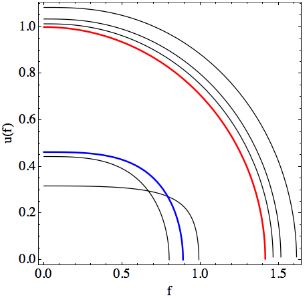

Figure 3:

Lifshitz solutions

We consider Lifshitz solution with $1 < z < 2$; The series expansion gives

$$u(f) = u_0 + \frac{(z-1)u_0}{2z^3(2-z)} f^{2z} + ... + bf^{z+2} + ...$$

Since the stationary equation is 2nd order ODE, the solution should include two parameters of solutions (u_0, b).

$$S \approx \frac{z}{2\pi\alpha'} \int \frac{d\sigma}{\sigma} \int \frac{df}{f^{z+1}}. \tag{9}$$

For $z < 2$,

$$S \sim \frac{\sqrt{z-1}}{\sqrt{\epsilon}},$$

while for $z > 2$,

$$S \sim \frac{1}{\epsilon^{\frac{z}{z+2}}}.$$

Lifshitz solutions

There are three features can be observed from series expansion above.

- The coefficient of f^{2z} term is positive if $1 < z < 2$; This means the minimal surface near the Lifshitz boundary bends outward when getting into the bulk.
- The coefficient of f^{2z} term vanishes at $z = 1$. This coincides with series expansion at $z = 1$. It raises a puzzle: how can we consider
$$u = \sqrt{1 - \frac{f^2}{2}}$$
as a limit of some Lifshitz solution at $z = 1$?
- The coefficient of f^{2z} diverges at $z = 2$.

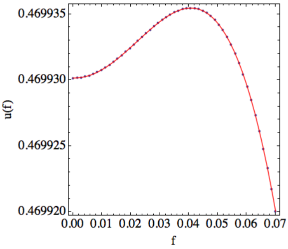

Figure 4:

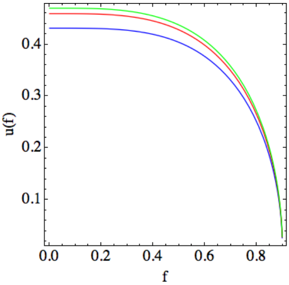

Figure 5:

Converge to AdS limit

In order to consider

$$u = \sqrt{1 - \frac{f^2}{2}}$$

as a limit of Lifshitz solution when $z \to 1$, we would have to find some general terms in the series expansion:

$$u(f) = u_0 + \sum_{i,j=1}^{\infty} A_{ij} f^{a_{ij}} + \sum_{m,n=1}^{\infty} B_{mn} f^{b_{mn}}, \tag{10}$$

$$a_{ij} = (2z-2)i + 2 + 2z(j-1) \tag{11}$$

$$b_{mn} = (2z-2)m + 4 - z + 2z(n-1), \tag{12}$$

Converge to AdS limit

$$A_{11} = \frac{(z-1)u_0}{2z^3(2-z)}; \qquad B_{11} = b \tag{13}$$

$$A_{21} = \frac{(z-1)u_0^3}{2z^4(2z-1)(3z-4)(z-2)}; \qquad B_{21} = -\frac{bu_0^2(2+z)}{6z^3} \tag{14}$$

$$A_{31} = \frac{(z-1)(2z-3)u_0^5}{2z^6(z-2)(5z-6)(3z-4)(3z-2)} \tag{15}$$

$$A_{k1} = \frac{u_0^{2k-1}(z-1)\prod_{\alpha=3}^{k}[(2\alpha-4)z-(2\alpha-3)]}{2z^{2k}[kz-(k-1)]\prod_{\beta=1}^{k}[(2\beta-1)z-2\beta]} \qquad (k \geq 3) \tag{16}$$

$$A_{12} = \frac{(z-1)(2z-1)(3z-1)u_0}{8z^6(3z-2)(2-z)} \tag{17}$$

Converge to AdS limit

An observation of equation (10) is at the $z = 1$ limit, many generically distinguished powers coincides. The coefficient of f^2 term receive contribution from $f^{2z}, f^{4z-2}, f^{6z-4}$... etc. Restrict ourselves near the light-like limit $u_0 = 1 + q\epsilon$

Summation

$$A_1 = \sum_{k=1}^{\infty} A_{k1} = \frac{\epsilon}{2}(1 - 2\epsilon)\frac{u_0}{1 - u_0^2} = -\frac{1}{4q} + \mathcal{O}(\epsilon),$$

This is a finite number. To get the AdS solution, we need $q = 1$.

Conclusion and Future work

- We use A-M trick to calculate the timelike Wilson loop in Lifshitz spacetime. Surprisingly, the divergence at $1 < z < 2$ case is universal and stronger than AdS.
- All of the surface we found so far are timelike. To get spacelike surface, one would need to hit a singularity in the bulk. Relaxation methods fail. The singularity looks exactly the momentum barrier in Lifshitz spacetime (1308.5689). It might interesting to work the interpretation.
- We don't have physical understanding of outward bending behavior of minimal surface at the boundary.
- There are some specific values of $z = \frac{2n}{2n-1}$, in which there are some A_{k1} diverge. These branches would have log terms. Are there any physical meanings for these values?