Baby Skyrme models without a potential term

Jenny Ashcroft

SMSAS, University of Kent

17th December 2014

Outline

1. Motivation
   - Skyrme Models
   - Derrick's theorem
   - Baby Skyrme models without a potential
2. Results
   - Overview
   - The $\alpha = 1$ model
   - The $\alpha = 1/2$ model
3. Summary

The Baby Skyrme Model

The static energy of the baby Skyrme model is given by

$$E_{BS} = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\partial_i\phi \cdot \partial_i\phi + \frac{1}{4}|\partial_i\phi \times \partial_j\phi|^2 + m^2(1-\phi_3) \right\} \mathrm{d}^2x,$$

where m is a mass term and the field $\phi : \mathbb{R}^2 \to S^2$ is the three-component vector $\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^\intercal$.

Static baby Skyrmions

- are energetic minima of this energy,
- satisfy $\phi \cdot \phi = 1$,
- have finite energy - require ϕ constant at spatial boundary so

$$\lim_{|x|\to\infty} \phi = (0,0,1),$$

- have topological charge B, an integer winding number describing net number of solitons given by

$$B = \frac{1}{4\pi} \int_{\mathbb{R}^2} \phi \cdot (\partial_1\phi \times \partial_2\phi)\ \mathrm{d}^2x.$$

The Skyrme Model

The baby Skyrme model is a (2+1)-dimensional analogue of the (3+1)-dimensional Skyrme model

Static energy of the Skyrme model is given by

$$E_S = \frac{1}{12\pi^2}\int_{\mathbb{R}^3}\left\{-\frac{1}{2}\mathrm{Tr}(R_iR_i) - \frac{1}{16}\mathrm{Tr}([R_i,R_j][R_i,R_j]) + m^2\mathrm{Tr}(1-U)\right\}\mathrm{d}^3x,$$

where m is related to the pion-mass, $U:\mathbb{R}^3 \to \mathrm{SU}(2)$ and $R_i = (\partial_i U)U^\dagger$.

Topological solitons called Skyrmions

- can be interpreted as baryons
- have integer topological charge B called baryon number

Derrick's Theorem

An important non-existence theorem for field theories defined in flat space.

Derrick's theorem

Suppose that for an arbitrary finite energy field configuration $\Psi(x)$, which is not the vacuum, the function $e(\mu)$, denoting the energy of the field configuration after applying the spatial rescaling $x \mapsto \mu x$, has no stationary point. Then the theory has no static solutions of the field equation with finite energy, other than the vacuum.

Derrick's theorem

Baby Skyrme model

$$E_{BS} = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \partial_i \phi \cdot \partial_i \phi + \frac{1}{4} |\partial_i \phi \times \partial_j \phi|^2 + m^2 (1 - \phi_3) \right\} \mathrm{d}^2 x = E_2 + E_4 + E_0.$$

Apply rescaling $x \mapsto \mu x$, $\mu > 0$,

$$e_{BS}(\mu) = (\mu^2 E_2 + \mu^4 E_4 + E_0)\mu^{-2} = E_2 + \mu^2 E_4 + \mu^{-2} E_0.$$

- Need E_0 to have solitons.

Skyrme model

After rescaling

$$e_S(\mu) = (\mu^2 E_2 + \mu^4 E_4 + E_0)\mu^{-3} = \mu^{-1} E_2 + \mu E_4 + \mu^{-3} E_0.$$

- Do not need E_0 to have solitons.

Deriving new models

The static energy functional for our models is

$$E = \int_{\mathbb{R}^2} \left\{ (\partial_i \phi \cdot \partial_i \phi)^{\alpha} + (|\partial_i \phi \times \partial_j \phi|^2)^{\beta} \right\} \mathrm{d}^2 x,$$

for $\alpha,\ \beta \in \mathbb{R}$.
Under the rescaling action $x \mapsto \mu x$, this becomes

$$e(\mu) = \mu^{2\alpha-2} E_2 + \mu^{4\beta-2} E_4.$$

So to evade Derrick's theorem we choose

$$\alpha < 1 \ \text{ and } \ \beta > \frac{1}{2},$$

$$\text{or } \ \alpha = 1 \ \text{ and } \ \beta = \frac{1}{2}.$$

Energy Bounds

Define a matrix D by $D_{ij} = \partial_i \phi \cdot \partial_j \phi$. We can rewrite the energy as

$$E = \int_{\mathbb{R}^2} \left\{ (\lambda_1^2 + \lambda_2^2)^\alpha + (2\lambda_1^2 \lambda_2^2)^\beta \right\} \mathrm{d}^2 x,$$

where $\lambda_1^2,\ \lambda_2^2$ are the eigenvalues of D.

We can write

$$|B| = \frac{1}{4\pi} \left| \int \lambda_1 \lambda_2 \ \mathrm{d}^2 x \right|.$$

Useful inequality

For a, b non-negative, it holds that

$$\frac{a+b}{2} \geq \sqrt{ab},$$

with equality if and only if $a = b$.

Energy Bounds

$$E = 2\int_{\mathbb{R}^2} \left\{ \frac{1}{2}(\lambda_1^2+\lambda_2^2)^\alpha + \frac{1}{2}(2\lambda_1^2\lambda_2^2)^\beta \right\} \mathrm{d}^2x$$

$$\geq 2\int_{\mathbb{R}^2} (\lambda_1^2+\lambda_2^2)^{\frac{\alpha}{2}} (2\lambda_1^2\lambda_2^2)^{\frac{\beta}{2}} d^2x$$

$$\geq 2\int_{\mathbb{R}^2} 2^{\frac{\alpha}{2}} (\lambda_1\lambda_2)^{\frac{\alpha}{2}} 2^{\frac{\beta}{2}} |\lambda_1\lambda_2|^\beta d^2x$$

$$= 2^{1+\frac{\alpha+\beta}{2}} \int_{\mathbb{R}^2} |\lambda_1\lambda_2|^{\frac{\alpha+2\beta}{2}} d^2x.$$

We want a linear bound in terms of the topological charge, so choose

$$\beta = 1 - \frac{\alpha}{2}.$$

This leads to the bound

$$E \geq 2^{\frac{3}{2}+\frac{\alpha}{4}} 4\pi|B|.$$

Axially Symmetric solutions

We substitute into the energy functional the ansatz

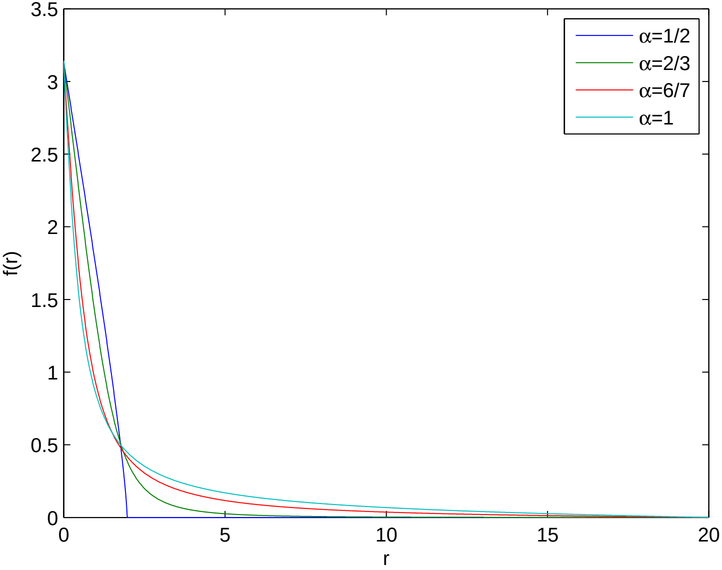
$$\phi(x) = (\sin f(r) \cos B\theta,\ \sin f(r) \sin B\theta,\ \cos f(r)),$$

obtaining a new energy which only depends on r

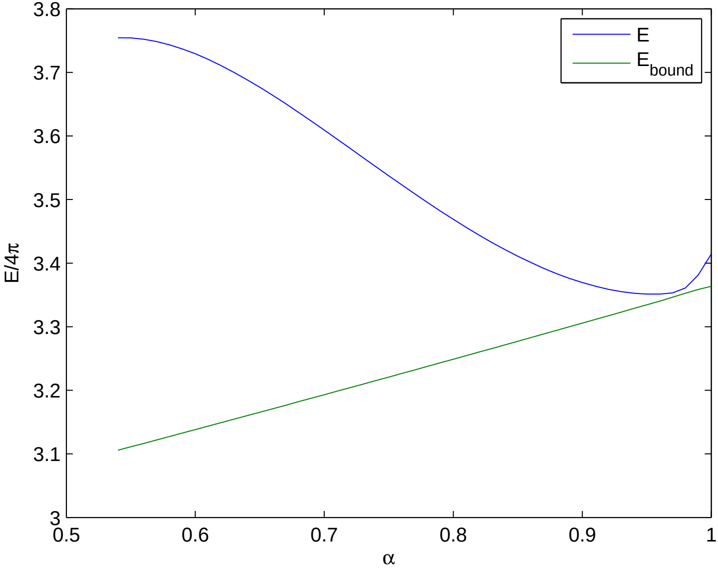
$$E = 2\pi \int \left((f'^2 + \frac{B^2}{r^2} \sin^2 f)^\alpha + (2f'^2 \frac{B^2}{r^2} \sin^2 f)^\beta \right) r\ dr.$$

We can calculate the equations of motion for this and solve them numerically for $f(r)$ subject to $f(0) = \pi$, $f(\infty) = 0$, to obtain axially symmetric solutions.

Charge 1 Profile Functions



Charge 1 Energy



The $\alpha = 1$ model

Static energy

$$E = \int_{\mathbb{R}^2} \left\{ (\partial_i \phi \cdot \partial_i \phi) + \sqrt{|\partial_i \phi \times \partial_j \phi|^2} \right\} \mathrm{d}^2 x.$$

This model is scale invariant

$$e(\mu) = \mu^{2\cdot 1 - 2} E_2 + \mu^{4\cdot \frac{1}{2} - 2} E_4 = E_2 + E_4.$$

Energy bound

By a completing the square argument, the model can be shown to satisfy the following energy bound

$$E \geq 4\pi(2 + \sqrt{2})|B| > 2^{\frac{7}{4}} \cdot 4\pi |B|.$$

For profile functions, this bound is attained for solutions of

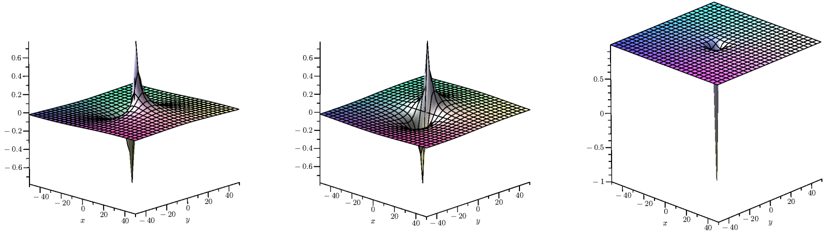
$$f' = -\frac{B}{r} \sin f.$$

Solutions

Set $R = r^B e^{iB\theta} = z^B$, then

$$\phi = \frac{1}{1+|R|^2}\Big(R+\bar{R},\ -i(R-\bar{R}),\ |R|^2-1\Big).$$

Charge 1 solution



ϕ_1 ϕ_2 ϕ_3

The $\alpha = 1/2$ model

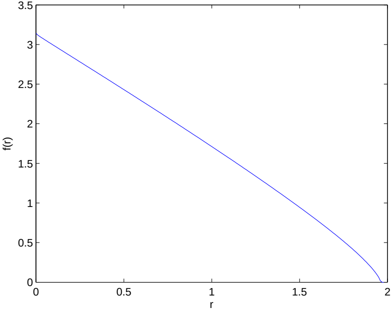
Static energy

$$E = \int_{\mathbb{R}^2} \left\{ \sqrt{(\partial_i \phi \cdot \partial_i \phi)} + (|\partial_i \phi \times \partial_j \phi|^2)^{\frac{3}{4}} \right\} \mathrm{d}^2 x.$$

Rescaling is the same as for the Skyrme model

$$e(\mu) = \mu^{2\cdot\frac{1}{2}-2} E_2 + \mu^{4\cdot\frac{3}{4}-2} E_4 = \mu^{-1} E_2 + \mu E_4.$$

Compacton solutions



Summary

- To evade Derrick's theorem, it is necessary for the baby Skyrme model to have a potential term but this is not the case for the Skyrme model
- We have developed new baby Skyrme models which do not require a potential to have topological solitons
- The new models satisfy a topological energy bound
- Solutions show a range of behaviours including
  - Compactons in $\alpha = \frac{1}{2}$ model
  - Exact solutions in $\alpha = 1$ model