Non-associative geometry in flux compactifications of string theory

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Overview

Motivation

Work in progress/ outlook

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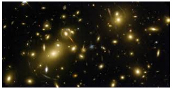
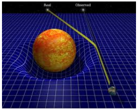
Light and the nature of space-time

Light follows the geodesics of space-time



Gravitational lensing



Massive objects curve space-time in their vicinity

What is the nature of quantum space-time?

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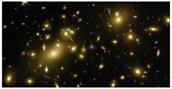

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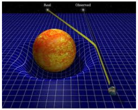

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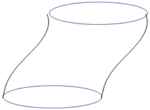
What is the nature of quantum space-time?

Closed strings probe or ‘feel-out’ space-time on the quantum level ($\sim 10^{-35} m$)



Worldsheet of closed string probing space-time

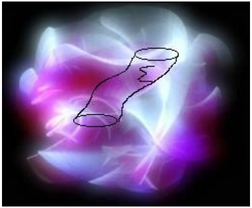
6 unobserved dimensions of strings' 10 dimensional target space are perhaps rolled up/ compactified in

Flux compactifications

- string vacua with p-form fluxes along the extra dimensions

Motivation

Flux compactifications of *closed* string theory



$$\boxed{X : \Sigma \longrightarrow M = \mathbb{R}^4 \times K_H}$$

H-flux, $H = \mathrm{d}\, B$, turned on in extra dimensions of string vacua K_H

Motivation

Non-commutative and non-associative space-time geometry

$$\boxed{\text{geometric } K_H \overset{T-\text{duality}}{\rightsquigarrow} \text{“non} - \text{geometric” } K_R}$$

- closed strings propagating and winding in the R-flux background probe a non-commutative and non-associative space-time geometry (Blumenhagen, Plauschinn: 2010, Lüst: 2010)
- confirmed by explicit string and CFT calculations (Blumenhagen, Deser, Lüst, Plauschinn, Rennecke: 2011, Condeescu, Florakis, Lüst: 2012)

Constant trivector R-flux: $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$

Coordinate algebra probed by closed strings in R-flux compactification:

non-commutative $[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \ , \quad [x^i, \partial_j] = i\hbar\delta^i{}_j \quad \text{and} \quad [\partial_i, \partial_j] = 0$

non-associative $[x^i, x^j, x^k] = \ell_s^4 R^{ijk}$

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Attempt to understand non-geometric space-time

- Kontsevich's deformation quantization of twisted Poisson manifolds provides explicit star product realizations of this non-associative geometry (Mylonas, Schupp, Szabo: 2012)

- If one replaces

$$\boxed{x^i \cdot x^j \longmapsto x^i \star x^j}$$

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- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

$\star$-product: $\star = \mu \circ F^{-1}$

flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright (x^j \star x^i)$

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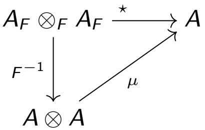
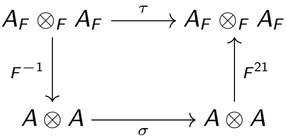
Motivation

Where these formulae come from...

$\star$ and flip:





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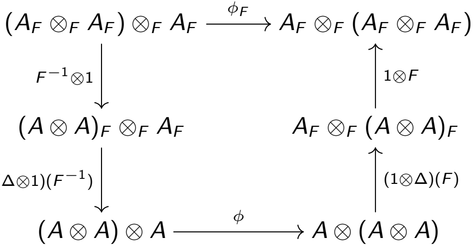

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Goal Mathematical development of a framework to describe a large class of non-commutative and non-associative geometries, including the non-geometric flux compactification above.

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Gelfand-Naimark "Manifolds can be analyzed by studying functions on them."

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

$$G\text{-Man}^{\mathrm{op}} \xrightarrow[\simeq]{C^\infty} {}^{U\mathfrak{g}}\mathrm{Alg}$$

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation

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"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

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- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

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$$F = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\left(\tfrac{1}{4}\,\mathrm{R}^{ijk}\,(m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\right)\right)$$

- The star product is given by $\star = \mu \circ F^{-1}$, which yields on A

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Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

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Serre-Swan "Vector bundles may be analysed by studying their modules of sections."

Lem M a manifold with G-action. Then there is a functor:

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$$ {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{F} {}^{H}{}_{A}\mathscr{M}_{A}$$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

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- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in^{H}$ Alg we obtain:

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Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A$ $\rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
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Overview

Motivation

Work in progress/ outlook

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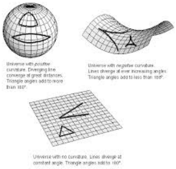
- Differential operators, connections, Riemannian geometry in ${}^{H}{}_{A}\mathscr{M}_{A}$
- Develop a gravity theory in ${}^{H}{}_{A}\mathscr{M}_{A}$ which is a candidate for a low-energy effective theory for non-geometric closed string theory



Geometry on curved spaces

Thank you