Non-associative geometry in flux compactifications of string theory

Gwendolyn E. Barnes

Department of Mathematics, Heriot-Watt University, Edinburgh, United Kingdom

Maxwell Institute for Mathematical Sciences

The Tait Institute

Joint work with A. Schenkel and R. J. Szabo [arXiv:1409.6331]
to appear in Journal of Geometry and Physics

December 16, 2014

Overview

Motivation

Non-commutative and non-associative algebras from deformations

Non-commutative and non-associative bundles from deformations

Work in progress/ outlook

Overview

Motivation

Non-commutative and non-associative algebras from deformations

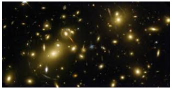
Non-commutative and non-associative bundles from deformations

Work in progress/ outlook

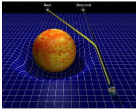
Motivation

Light and the nature of space-time

Light follows the geodesics of space-time



Gravitational lensing



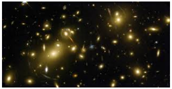
Massive objects curve space-time in their vicinity

What is the nature of quantum space-time?

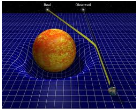
Motivation

Light and the nature of space-time

Light follows the geodesics of space-time



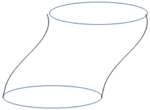
Gravitational lensing



Massive objects curve space-time in their vicinity

What is the nature of quantum space-time?

Closed strings probe or 'feel-out' space-time on the quantum level ($\sim 10^{-35} m$)



Worldsheet of closed string probing space-time

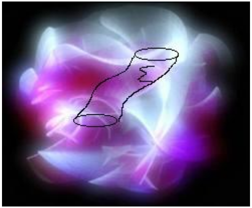
6 unobserved dimensions of strings' 10 dimensional target space are perhaps rolled up/ compactified in

Flux compactifications

- string vacua with p-form fluxes along the extra dimensions

Motivation

Flux compactifications of *closed* string theory



$$\boxed{X : \Sigma \longrightarrow M = \mathbb{R}^4 \times K_H}$$

H-flux, $H = \mathrm{d}\,B$, turned on in extra dimensions of string vacua K_H

Motivation

Non-commutative and non-associative space-time geometry

$$\text{geometric } K_H \overset{T-\text{duality}}{\rightsquigarrow} \text{“non} - \text{geometric”} \;\; K_R$$

- closed strings propagating and winding in the R-flux background probe a non-commutative and non-associative space-time geometry (Blumenhagen, Plauschinn: 2010, Lüst: 2010)
- confirmed by explicit string and CFT calculations (Blumenhagen, Deser, Lüst, Plauschinn, Rennecke: 2011, Condeescu, Florakis, Lüst: 2012)

Constant trivector R-flux: $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_i \wedge \partial_k$

Coordinate algebra probed by closed strings in R-flux compactification:

non-commutative $[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \;, \quad [x^i, \partial_j] = i\hbar \delta^i{}_j \;$ and $\; [\partial_i, \partial_j] = 0$

non-associative $[x^i, x^j, x^k] = \ell_s^4 R^{ijk}$

$$\boxed{\text{geometric } K_H \overset{T-\text{duality}}{\rightsquigarrow} \text{“non} - \text{geometric” } K_R}$$

- closed strings propagating and winding in the R-flux background probe a non-commutative and non-associative space-time geometry (Blumenhagen, Plauschinn: 2010, Lüst: 2010)
- confirmed by explicit string and CFT calculations (Blumenhagen, Deser, Lüst, Plauschinn, Rennecke: 2011, Condeescu, Florakis, Lüst: 2012)

Constant trivector R-flux: $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$

Coordinate algebra probed by closed strings in R-flux compactification:

non-commutative $[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \,, \quad [x^i, \partial_j] = i\hbar \delta^i{}_j \;\text{ and }\; [\partial_i, \partial_j] = 0$

non-associative $[x^i, x^j, x^k] = \ell_s^4 R^{ijk}$

Motivation

Non-commutative and non-associative space-time geometry

$$\text{geometric } K_H \overset{T-\text{duality}}{\rightsquigarrow} \text{“non} - \text{geometric” } K_R$$

- closed strings propagating and winding in the R-flux background probe a non-commutative and non-associative space-time geometry (Blumenhagen, Plauschinn: 2010, Lüst: 2010)
- confirmed by explicit string and CFT calculations (Blumenhagen, Deser, Lüst, Plauschinn, Rennecke: 2011, Condeescu, Florakis, Lüst: 2012)

Constant trivector R-flux: $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$

Coordinate algebra probed by closed strings in R-flux compactification:

non-commutative $[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \ , \quad [x^i, \partial_j] = i\hbar \delta^i{}_j \ \text{ and } \ [\partial_i, \partial_j] = 0$

non-associative $[x^i, x^j, x^k] = \ell_s^4 R^{ijk}$

Motivation

Attempt to understand non-geometric space-time

- Kontsevich's deformation quantization of twisted Poisson manifolds provides explicit star product realizations of this non-associative geometry (Mylonas, Schupp, Szabo: 2012)

- If one replaces

$$\boxed{x^i \cdot x^j \longmapsto x^i \star x^j}$$

one recovers the "non-geometric" commutation relations and Jacobiator

$$\text{nc} \quad [x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \,, \quad [x^i, \partial_j]_\star = i\hbar \delta^i{}_j \quad \text{and} \quad [\partial_i, \partial_j]_\star = 0$$

$$\text{na} \quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- **The coordinate algebra with the $\star$-product is a non-commutative and non-associative algebra on the R-flux compactification**

- Kontsevich's deformation quantization of twisted Poisson manifolds provides explicit star product realizations of this non-associative geometry (Mylonas, Schupp, Szabo: 2012)

- If one replaces

$$\boxed{x^i \cdot x^j \longmapsto x^i \star x^j}$$

one recovers the "non-geometric" commutation relations and Jacobiator

$$\text{nc} \quad [x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \ , \quad [x^i, \partial_j]_\star = i\hbar \delta^i{}_j \ \text{ and } \ [\partial_i, \partial_j]_\star = 0$$

$$\text{na} \quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- The coordinate algebra with the $\star$-product is a non-commutative and non-associative algebra on the R-flux compactification

Motivation

- Kontsevich's deformation quantization of twisted Poisson manifolds provides explicit star product realizations of this non-associative geometry (Mylonas, Schupp, Szabo: 2012)

- If one replaces

$$\boxed{x^i \cdot x^j \longmapsto x^i \star x^j}$$

  one recovers the "non-geometric" commutation relations and Jacobiator

$$\text{nc} \quad [x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \ , \quad [x^i, \partial_j]_\star = i\hbar \delta^i{}_j \ \text{ and } \ [\partial_i, \partial_j]_\star = 0$$

$$\text{na} \quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- **The coordinate algebra with the $\star$-product is a non-commutative and non-associative algebra on the R-flux compactification**

Motivation

Twist deformation quantisation

- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

$\star$-product: $\star = \mu \circ F^{-1}$

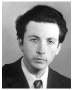
flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright (x^j \star x^i)$

associator: $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

$$(x^i \star x^j) \star x^k = \phi_F \triangleright (x^j \star (x^i \star x^k))$$

- **quasi-Hopf algebra** (H, τ, ϕ_F) = **"generalised quantum group / quantum symmetries"**

Vladimir Gershonovich Drinfeld



- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

  $\star$-product: $\star = \mu \circ F^{-1}$

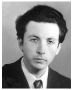
  flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright (x^j \star x^i)$

  associator: $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

  $(x^i \star x^j) \star x^k = \phi_F \triangleright (x^j \star (x^i \star x^k))$

- **quasi-Hopf algebra (H, τ, ϕ_F) = "generalised quantum group / quantum symmetries"**

Vladimir Gershonovich Drinfeld



Motivation

Twist deformation quantisation

- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

$\star$-product: $\star = \mu \circ F^{-1}$

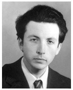
flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright (x^j \star x^i)$

associator: $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

$$(x^i \star x^j) \star x^k = \phi_F \triangleright (x^j \star (x^i \star x^k))$$

- **quasi-Hopf algebra (H, τ, ϕ_F) = "generalised quantum group / quantum symmetries"**

Vladimir Gershonovich Drinfeld



Motivation

Twist deformation quantisation

- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

$\star$-product: $\star = \mu \circ F^{-1}$

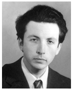
flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright (x^j \star x^i)$

associator: $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

$$(x^i \star x^j) \star x^k = \phi_F \triangleright (x^j \star (x^i \star x^k))$$

- **quasi-Hopf algebra (H, τ, ϕ_F) = "generalised quantum group / quantum symmetries"**

Vladimir Gershonovich Drinfeld



- (Mylonas, Schupp, Szabo: 2013) observed that noncommutative and nonassociative star products can be obtained via a cochain twisting of classical symmetries to a quasi-Hopf algebra

- For a particular choice of "classical algebra of symmetries" $\mathfrak{g}$
- and "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$, we obtain

$\star$-product: $\star = \mu \circ F^{-1}$

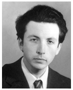
flip: $\tau = F^{21} \circ \sigma \circ F^{-1}$ $\quad x^i \star x^j = \tau \triangleright \left(x^j \star x^i \right)$

associator: $\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)$

$$(x^i \star x^j) \star x^k = \phi_F \triangleright \left(x^j \star (x^i \star x^k) \right)$$

- **quasi-Hopf algebra** $(H, \tau, \phi_F) =$ **"generalised quantum group / quantum symmetries"**

Vladimir Gershonovich Drinfeld



Motivation

Where these formulae come from...

$\star$ and flip:

$$\begin{array}{ccc} A_F \otimes A_F & \xrightarrow{\star} & A \\ \downarrow{\scriptstyle F^{-1}} & \nearrow{\scriptstyle \mu} & \\ A \otimes A & & \end{array} \qquad \begin{array}{ccc} A_F \otimes A_F & \xrightarrow{\tau} & A_F \otimes A_F \\ \downarrow{\scriptstyle F^{-1}} & & \uparrow{\scriptstyle F^{21}} \\ A \otimes A & \xrightarrow[\sigma]{} & A \otimes A \end{array}$$

$$\boxed{\star = \mu \circ F^{-1}} \qquad \boxed{\tau = F^{21} \circ \sigma \circ F^{-1}}$$

associator:

$$\begin{array}{ccc} (A_F \otimes A_F) \otimes A_F & \xrightarrow{\phi_F} & A_F \otimes (A_F \otimes A_F) \\ \downarrow{\scriptstyle F^{-1} \otimes 1} & & \uparrow{\scriptstyle 1 \otimes F} \\ (A \otimes A)_F \otimes A_F & & A_F \otimes (A \otimes A)_F \\ \downarrow{\scriptstyle (\Delta \otimes 1)(F^{-1})} & & \uparrow{\scriptstyle (1 \otimes \Delta)(F)} \\ (A \otimes A) \otimes A & \xrightarrow{\phi} & A \otimes (A \otimes A) \end{array}$$

$$\boxed{\phi_F = (1 \otimes F) \circ (1 \otimes \Delta)(F) \circ \phi \circ (\Delta \otimes 1)(F^{-1}) \circ (F^{-1} \otimes 1)}$$

Goal Mathematical development of a framework to describe a large class of non-commutative and non-associative geometries, including the non-geometric flux compactification above.

Overview

Israel Moiseevic Gelfand

' I have mentioned the closeness between the style of mathematics and the style of classical music or poetry. I was happy to find the following four common features: first – beauty, second – simplicity, third – exactness, fourth – crazy ideas. The combination of these four things: beauty, exactness, simplicity and crazy ideas is just the heart of mathematics, the heart of classical music. '

Mark Aronovich Naimark

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

Gelfand-Naimark $$G\text{-Man}^{\mathrm{op}} \xrightarrow{C^\infty} {}^{U\mathfrak{g}}\mathrm{Alg}$$

"Manifolds may be analyzed by studying functions on them."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation $${}^{U\mathfrak{g}}\mathrm{Alg} \xrightarrow[\simeq]{F} {}^{H}\mathrm{Alg}$$

"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

Application Recover MSS Algebra by choosing $U\mathfrak{g}$, a particular algebra A in ${}^{U\mathfrak{g}}\mathrm{Alg}$ and twist $F \in U\mathfrak{g} \otimes U\mathfrak{g}$.

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

Gelfand-Naimark $$G\text{-Man}^{\mathrm{op}} \xrightarrow{\;C^\infty\;} {}^{U\mathfrak{g}}\mathrm{Alg}$$

"Manifolds may be analyzed by studying functions on them."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation $${}^{U\mathfrak{g}}\mathrm{Alg} \xrightarrow[\simeq]{\;F\;} {}^{H}\mathrm{Alg}$$

"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

Application Recover MSS Algebra by choosing $U\mathfrak{g}$, a particular algebra A in ${}^{U\mathfrak{g}}\mathrm{Alg}$ and twist $F \in U\mathfrak{g} \otimes U\mathfrak{g}$.

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

Gelfand-Naimark $$G\text{-Man}^{\mathrm{op}} \xrightarrow{\quad C^\infty \quad} {}^{U\mathfrak{g}}\mathrm{Alg}$$

"Manifolds may be analyzed by studying functions on them."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation $${}^{U\mathfrak{g}}\mathrm{Alg} \xrightarrow[\simeq]{\quad F \quad} {}^{H}\mathrm{Alg}$$

"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

Application Recover MSS Algebra by choosing $U\mathfrak{g}$, a particular algebra A in ${}^{U\mathfrak{g}}\mathrm{Alg}$ and twist $F \in U\mathfrak{g} \otimes U\mathfrak{g}$.

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

Gelfand-Naimark
$$G\text{-Man}^{\mathrm{op}} \xrightarrow{\quad C^\infty \quad} {}^{U\mathfrak{g}}\mathrm{Alg}$$

"Manifolds may be analyzed by studying functions on them."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation
$${}^{U\mathfrak{g}}\mathrm{Alg} \xrightarrow[\simeq]{\quad F \quad} {}^{H}\mathrm{Alg}$$

"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

Application Recover MSS Algebra by choosing $U\mathfrak{g}$, a particular algebra A in ${}^{U\mathfrak{g}}\mathrm{Alg}$ and twist $F \in U\mathfrak{g} \otimes U\mathfrak{g}$.

We are interested in obtaining nc/ na spaces by deforming classical manifolds with a symmetry group action G.

Lem G a Lie group, $U\mathfrak{g}$ the universal enveloping algebra of its associated Lie algebra $\mathfrak{g}$. Then there is a functor:

Gelfand-Naimark
$$G\text{-}\mathrm{Man}^{\mathrm{op}} \xrightarrow{\;C^\infty\;} {}^{U\mathfrak{g}}\mathrm{Alg}$$

"Manifolds may be analyzed by studying functions on them."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation
$${}^{U\mathfrak{g}}\mathrm{Alg} \xrightarrow[\simeq]{\;F\;} {}^{H}\mathrm{Alg}$$

"Algebras transforming under classical symmetries are twisted to nc/ na algebras transforming under quantum symmetries H."

Remark Twist deformation quantisation is an equivalence of categories.

Application Recover MSS Algebra by choosing $U\mathfrak{g}$, a particular algebra A in ${}^{U\mathfrak{g}}\mathrm{Alg}$ and twist $F \in U\mathfrak{g} \otimes U\mathfrak{g}$.

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \big(\mathrm{R}^{ijk}\big)_{i,j,k=1}^{n}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac{1}{4}\,\mathrm{R}^{ijk}\,(m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\big)\big)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

nc $[x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k \ , \quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \ \text{ and } \ [p_i, p_j]_\star = 0$

na $[x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\big(\frac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\big)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \big(\mathrm{R}^{ijk}\big)^n_{i,j,k=1}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac14\,\mathrm{R}^{ijk}\,(m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\big)\big)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

$$\text{nc}\quad [x^i, x^j]_\star = \tfrac{i\ell_s^4}{3\hbar} R^{ijk} p_k \ , \quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \quad \text{and} \quad [p_i, p_j]_\star = 0$$

$$\text{na}\quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\big(\tfrac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\big)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \left(\mathrm{R}^{ijk}\right)^n_{i,j,k=1}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\left(\tfrac{1}{4}\,\mathrm{R}^{ijk}\left(m_{ij} \otimes t_k - t_i \otimes m_{jk}\right) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\right)\right)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

$$\text{nc} \quad [x^i, x^j]_\star = \tfrac{i\ell_s^4}{3\hbar} R^{ijk} p_k \ , \quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \quad \text{and} \quad [p_i, p_j]_\star = 0$$

$$\text{na} \quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\left(\tfrac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\right)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \big(\mathrm{R}^{ijk}\big)_{i,j,k=1}^n$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac14\,\mathrm{R}^{ijk}\,(m_{ij}\otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\big)\big)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

nc $[x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k\ , \quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \quad \text{and} \quad [p_i, p_j]_\star = 0$

na $[x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\big(\frac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\big)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j \, t_k - \delta^i{}_k \, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \big(\mathrm{R}^{ijk}\big)_{i,j,k=1}^{n}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,\big(\tfrac{1}{4}\,\mathrm{R}^{ijk}\,(m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\big)\big)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

$$\text{nc} \quad [x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k \ , \quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \quad \text{and} \quad [p_i, p_j]_\star = 0$$

$$\text{na} \quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\big(\frac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\big)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \left(\mathrm{R}^{ijk}\right)^n_{i,j,k=1}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\left(\tfrac{1}{4}\,\mathrm{R}^{ijk}\left(m_{ij} \otimes t_k - t_i \otimes m_{jk}\right) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i\right)\right)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

nc $\quad [x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k \;,\quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \;\text{ and }\; [p_i, p_j]_\star = 0$

na $\quad [x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$

- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\left(\frac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\right)$

Example: "Moyal-Weyl" analogue of non-associative algebra

- $\mathfrak{g}$ the non-Abelian nilpotent Lie algebra over $\mathbb{C}$ with generators $\{t_i, \tilde{t}^{\,i}, m_{ij} : 1 \leq i < j \leq n\}$ and Lie bracket relations

$$[\,\tilde{t}^{\,i}, m_{jk}] = \delta^i{}_j\, t_k - \delta^i{}_k\, t_j$$

- "classical algebra of symmetries" $U\mathfrak{g}$
- "R-flux": rank-three skew-symmetric real-valued tensor $\mathrm{R} = \left(\mathrm{R}^{ijk}\right)^n_{i,j,k=1}$
- "cochain twist" $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$F = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\left(\tfrac{1}{4}\,\mathrm{R}^{ijk}\left(m_{ij}\otimes t_k - t_i \otimes m_{jk}\right) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i}\otimes t_i\right)\right)$$

- The star product is given by $\star = \mu \circ F^{-1}$. On $A = C^\infty(\mathbb{R}^{2n})$, this yields

  nc $\;[x^i, x^j]_\star = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k \;,\quad [x^i, p_j]_\star = i\hbar\delta^i{}_j \;$ and $\; [p_i, p_j]_\star = 0$

  na $\;[x^i, x^j, x^k]_\star = \ell_s^4 R^{ijk}$
- The flip is given by $\tau = F^{-2}$
- The associator is given by $\phi_F = \exp\left(\frac{\hbar^2}{2}\,\mathrm{R}^{ijk}\, t_i \otimes t_j \otimes t_k\right)$

Overview

Motivation

Non-commutative and non-associative algebras from deformations

Non-commutative and non-associative bundles from deformations

Work in progress/ outlook

Jean-Pierre Albert Achille Serre

Richard Gordon Swan

Given a nc/ na space, we want to understand **all** H-equivariant vector bundles (e.g. tangent bundle, cotangent bundle) and operations between them

Lem M a manifold with G-action. Then there is a functor:

Serre-Swan $G\text{-VecBun}_M \xrightarrow{\Gamma^\infty} {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)}$

"Vector bundles may be analysed by studying their modules of sections."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation ${}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{F} {}^{H}{}_{A}\mathscr{M}_{A}$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

Given a nc/ na space, we want to understand **all** H-equivariant vector bundles (e.g. tangent bundle, cotangent bundle) and operations between them

Lem M a manifold with G-action. Then there is a functor:

Serre-Swan

$$G\text{-VecBun}_M \xrightarrow{\Gamma^\infty} {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)}$$

"Vector bundles may be analysed by studying their modules of sections."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation

$$ {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{F} {}^{H}{}_{A}\mathscr{M}_{A}$$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

Given a nc/ na space, we want to understand **all** H-equivariant vector bundles (e.g. tangent bundle, cotangent bundle) and operations between them

Lem M a manifold with G-action. Then there is a functor:

Serre-Swan

$$G\text{-VecBun}_M \xrightarrow{\Gamma^\infty} {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)}$$

"Vector bundles may be analysed by studying their modules of sections."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation

$${}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{F} {}^{H}{}_{A}\mathscr{M}_{A}$$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

Given a nc/ na space, we want to understand **all** H-equivariant vector bundles (e.g. tangent bundle, cotangent bundle) and operations between them

Lem M a manifold with G-action. Then there is a functor:

Serre-Swan

$$G\text{-VecBun}_M \xrightarrow{\quad\Gamma^\infty\quad} {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)}$$

"Vector bundles may be analysed by studying their modules of sections."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation

$${}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{\quad F\quad} {}^{H}{}_{A}\mathscr{M}_{A}$$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

Given a nc/ na space, we want to understand **all** H-equivariant vector bundles (e.g. tangent bundle, cotangent bundle) and operations between them

Lem M a manifold with G-action. Then there is a functor:

Serre-Swan
$$G\text{-VecBun}_M \xrightarrow{\Gamma^\infty} {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)}$$

"Vector bundles may be analysed by studying their modules of sections."

Thm F a twist of $U\mathfrak{g}$. Then there is a functor:

Quantisation
$$ {}^{U\mathfrak{g}}{}_{C^\infty(M)}\mathscr{M}_{C^\infty(M)} \xrightarrow[\simeq]{F} {}^{H}{}_{A}\mathscr{M}_{A}$$

"Modules of sections over classical algebras are twisted to nc/ na modules of sections over quantum algebras."

Remark Twist deformation quantisation is an equivalence of categories

Application Applied to our non-geometric space, this gives us all H-equivariant vector bundles over the nc/na algebra describing the flux compactification.

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in^{H}$ Alg we obtain:

  Thm ${}^{H}{}_{A}\mathscr{M}_{A}$ is a closed braided monoidal category $(\otimes_A, \tau_A, \mathrm{hom}_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A$ $\rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. hom_A are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\text{Field} \longrightarrow 1-\text{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in^{H}$ Alg we obtain:

Thm ${}^{H}{}_{A}\mathscr{M}_{A}$ is a closed braided monoidal category $(\otimes_A, \tau_A, \hom_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A \rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. $\hom_A$ are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\text{Field} \longrightarrow 1-\text{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in^{H}$ Alg we obtain:

  Thm ${}^{H}{}_{A}\mathscr{M}_{A}$ is a closed braided monoidal category $(\otimes_A, \tau_A, \hom_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A$ $\rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. $\hom_A$ are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\text{Field} \longrightarrow 1-\text{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^H{}_A\mathscr{M}_A$ of H-equivariant vb over $A \in^H \mathrm{Alg}$ we obtain:

  Thm ${}^H{}_A\mathscr{M}_A$ is a closed braided monoidal category $(\otimes_A, \tau_A, \mathrm{hom}_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A \rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. hom_A are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\mathrm{Field} \longrightarrow 1-\mathrm{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^H{}_A\mathscr{M}_A$ of H-equivariant vb over $A \in^H$ Alg we obtain:

Thm ${}^H{}_A\mathscr{M}_A$ is a closed braided monoidal category $(\otimes_A, \tau_A, \hom_A)$

Physical relevance This gives standard operations on fields:

1 nc/na vector bundles can be tensored $\otimes_A \rightsquigarrow$ tensor fields

2 $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V$ $\rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors

3 $\hom_A$ are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

metric: $g : V\text{Field} \longrightarrow 1-\text{Forms}$

curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in {}^{H}$ Alg we obtain:

  Thm ${}^{H}{}_{A}\mathscr{M}_{A}$ is a closed braided monoidal category $(\otimes_A, \tau_A, \mathrm{hom}_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A \rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V \rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. hom_A are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\mathrm{Field} \longrightarrow 1 - \mathrm{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

- The representation category of any quasi-Hopf algebra is a closed braided monoidal category, which means that it has a tensor product, a braiding and internal homomorphisms.

- For the category ${}^{H}{}_{A}\mathscr{M}_{A}$ of H-equivariant vb over $A \in^{H}$ Alg we obtain:

  Thm ${}^{H}{}_{A}\mathscr{M}_{A}$ is a closed braided monoidal category $(\otimes_A, \tau_A, \mathrm{hom}_A)$

Physical relevance This gives standard operations on fields:

1. nc/na vector bundles can be tensored $\otimes_A \rightsquigarrow$ tensor fields
2. $V \otimes_A W \xrightarrow{\tau_A} W \otimes_A V \rightsquigarrow$ allows us to define symmetric and anti-symmetric tensors
3. hom_A are nc/na homomorphism bundles $\rightsquigarrow$ e.g. g, R

   metric: $g : V\mathrm{Field} \longrightarrow 1 - \mathrm{Forms}$

   curvature: $R : V \longrightarrow V \otimes_A \Omega^2$

Overview

Motivation

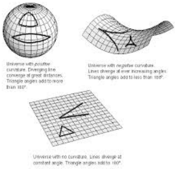
Non-commutative and non-associative algebras from deformations

Non-commutative and non-associative bundles from deformations

Work in progress/ outlook

Work in progress/ outlook

- Differential operators, connections, Riemannian geometry in ${}^{H}{}_{A}\mathscr{M}_{A}$
- Develop a gravity theory in ${}^{H}{}_{A}\mathscr{M}_{A}$ which is a candidate for a low-energy effective theory for non-geometric closed string theory



Geometry on curved spaces

Thank you