The $\mathcal{N} = 4$ Pentabox through Colour–Kinematics Duality

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Gustav Mogull
g.mogull@ed.ac.uk

Higgs Centre for Theoretical Physics, University of Edinburgh

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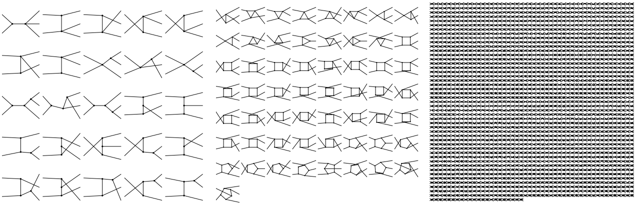
Complexity of Feynman Diagrams

- Scattering amplitudes are traditionally formed as a sum of constituent Feynman diagrams.
- These grow both in complexity and number with increasing numbers of scattered particles and internal loops.



(a) LO (b) NLO (c) NNLO

Figure: Three gluon jet production events.

- The situation is worse for gravity amplitudes as all possible kinds of vertex exist.

Hidden Structure in Yang Mills Amplitudes

- The Parke–Taylor formula for tree–level colour–ordered Yang Mills scattering amplitudes takes the form

$$A^{\text{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle},$$
$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma \in S_n} A^{\text{tree}}(\sigma(1), \sigma(2), \ldots, \sigma(n)) \text{Tr}\left[T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\right].$$

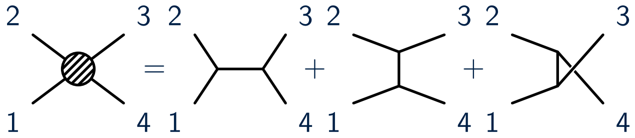
- Colour–ordered (colour–stripped) amplitudes are formed from planar Feynman diagrams with ordered external legs only.
- Recent advances in Yang Mills amplitudes are Lagrangian free, e.g. Yangian symmetry, Grassmannia, dual coordinates, the Amplituhedron, etc.
- Can we reconcile this elegant structure with our intuitive, yet computationally impractical, Feynman diagrams?

The Tree–Level Feynman Diagram Expansion

- The Feynman diagram expansion at tree–level is realised as a sum of cubic graphs only.

$$A_m^{\text{tree}} = g^{m-2} \sum_{\text{diagrams j}} \frac{c_j n_j}{\mathcal{D}_j}.$$

- $\mathcal{D}_j$ are products of Feynman propagators, c_j are colour factors (products of f^{abc}s) and n_j are kinematic numerators.
- At 4 points we have the s, t and u channels.



Colour–Kinematics Duality

- Colour–factors of diagrams often satisfy Jacobi identities of the form

$$c_i \pm c_j \pm c_k = 0,$$

due to $f^{abc} f^{cde} + f^{bcd} f^{ade} + f^{cad} f^{bde} = 0$.

- This occurs whenever three diagrams are the same, except for internal s, t and u– channels.
- At tree–level, it has been proven (arXiv:0805.3993) that we may choose Bern, Carrasco & Johansson (BCJ) kinematic numerators, n_i, satisfying

$$n_i \pm n_j \pm n_k = 0.$$

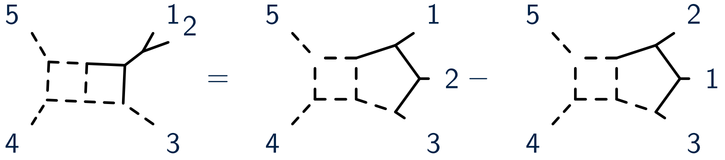
- Does this imply the existence of a kinematic group?

Loop–Level Expressions

- Existence of loop–level BCJ numerators is merely conjectured, though there is strong evidence.
- Yang Mills amplitudes take the form, with $D = 4 - 2\epsilon$,

$$\mathcal{A}_m^{\text{L-loop}} = i^L g^{m-2+2L} \sum_{\text{diagrams j}} \int \prod_{k=1}^{L} \frac{d^D \ell_k}{(2\pi)^D} \frac{1}{S_j} \frac{n_j(\ell_k) c_j}{\mathcal{D}_j(\ell_k)}.$$

- An example of a BCJ move on the “pentabox” numerator would be



The Double–Copy Formula

- This gives tree–level gravity amplitudes from BCJ numerators:

$$\mathcal{M}_m^{\text{tree}} = i\left(\frac{\kappa}{2}\right)^{m-2} \sum_j \frac{n_j \tilde{n}_j}{\mathcal{D}_j}.$$

- At loop–level, the double–copy formula generalises to

$$\mathcal{M}_m^{\text{L-loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_j \int \prod_{k=1}^{L} \frac{d^D \ell_k}{(2\pi)^D} \frac{1}{S_j} \frac{n_j(\ell_k)\tilde{n}_j(\ell_k)}{\mathcal{D}_j(\ell_k)}.$$

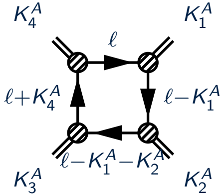
- These formulae continue to hold in the supersymmetric regime, potentially providing supergravity amplitudes at loop–level.

Finding BCJ Numerators

- BCJ systems are formed by diagrams making up an L-loop amplitude.
- Candidate numerators must satisfy 3 important properties:
  1. All possible BCJ moves of the form $n_i \pm n_j \pm n_k = 0$.
  2. Any symmetries of the corresponding graphs.
  3. Reproduction of the complete amplitude on summation of diagrams.
- It suffices to determine the numerators of the master diagrams: from these, all other numerators are straightforwardly obtainable through BCJ moves.
- However, if we need to compare to a known amplitude, what have we achieved?

Generalized Unitarity in $\mathcal{N} = 4$ at 1 Loop

- Colour–ordered, 1–loop $\mathcal{N} = 4$ amplitudes are expressible as a sum of box diagrams,



$$A^{\text{1–loop}}(1,2,\ldots,n) = \int \frac{d^4\ell}{(2\pi)^4} \mathcal{A}^{\text{1–loop}}(1,2,\ldots,n)$$

$$= \sum_{\text{channels } A} \int \frac{d^4\ell}{(2\pi)^4} \frac{B_A}{\ell^2(\ell-K_1^A)^2(\ell-K_1^A-K_2^A)^2(\ell+K_4^A)^2}.$$

- $\{K_i^A\}$ is an ordered partition of the external momenta p_i, e.g. $\{K_1, K_2, K_3, K_4\} = \{p_1+p_2, p_3+p_4, p_5, p_6\}$ at 6 points.
- The coefficients B_A are unknown and independent of ℓ.

Generalized Unitarity in $\mathcal{N} = 4$ at 1 Loop

$$\mathcal{A}(1,2,\ldots,n) = \sum_{\text{channels } C} \frac{B_C}{\ell^2(\ell - K_1^C)^2(\ell - K_1^C - K_2^C)^2(\ell + K_4^C)^2}$$

$$\ell^2(\ell - K_1^A)^2(\ell - K_1^A - K_2^A)^2(\ell + K_4^A)^2\mathcal{A}(1,2,\ldots,n)$$

$$= \sum_{\text{channels } C} B_C \frac{\ell^2(\ell - K_1^A)^2(\ell - K_1^A - K_2^A)^2(\ell + K_4^A)^2}{\ell^2(\ell - K_1^C)^2(\ell - K_1^C - K_2^C)^2(\ell + K_4^C)^2}.$$

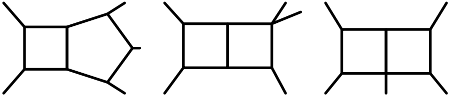
- Choose the 4 components of ℓ such that $\ell^2 = (\ell - K_1^A)^2 = (\ell - K_1^A - K_2^A)^2 = (\ell + K_4^A)^2 = 0$.

$$\begin{aligned} B_A &= \ell^2(\ell - K_1^A)^2(\ell - K_1^A - K_2^A)^2(\ell + K_4^A)^2\mathcal{A}(1,2,\ldots,n) \\ &= \mathrm{Cut}_A(1,2,\ldots,n). \end{aligned} \quad (1)$$

- Key point: a knowledge of the cuts suffices to reconstruct the full amplitude.

Generalizing to the 2–Loop, 5–Point System

- We need only compare to 3 different cuts, all of which contain no triangles, bubbles or tadpoles.



- We compare these to the BCJ expansion (rather than the irreducible expansion) of the colour–ordered integrand,

$$\mathcal{A}^{\text{2–loop}}(12345;\ell_1,\ell_2)=\sum_{\text{diagrams } i}\frac{n_i}{\mathcal{D}_i}.$$

- Nonplanar graphs and topologies make no contribution as we are interested in the colour–ordered amplitude.

The 5–Point, 2–Loop System in $\mathcal{N} = 4$

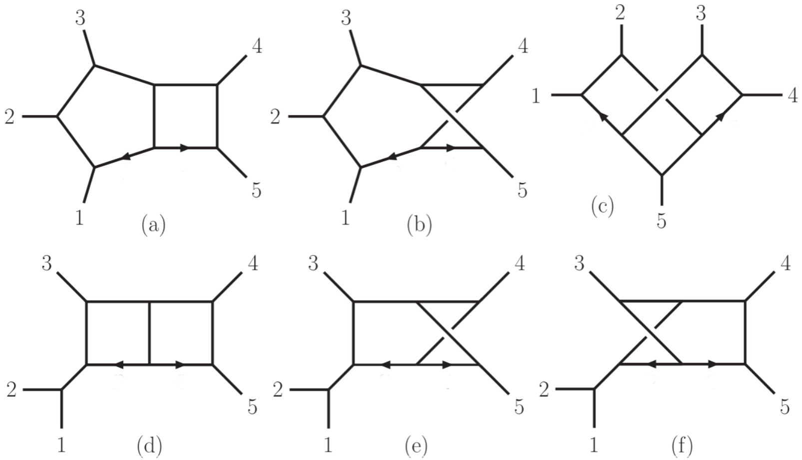
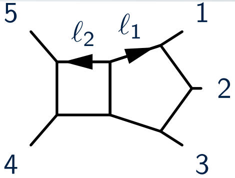


Figure: The 6 diagrams contributing to the 5–point, 2–loop amplitude in $\mathcal{N} = 4$.

The New Approach



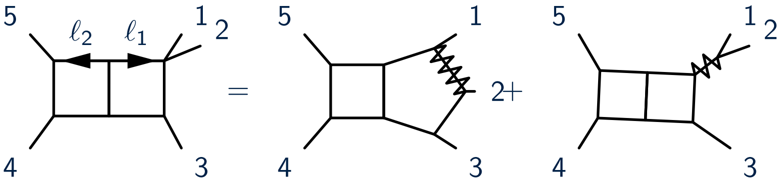
- $\Delta = \text{Cut}_{\text{pentabox}} = \ell_1^2(\ell_1 - p_1)^2(\ell_1 - p_1 - p_2)^2 \ldots (\ell_1 + \ell_2)^2 \mathcal{A}^{\text{2–loop}}$ is the maximal cut of the integrand, taken when all pentabox propagators are zero.
- The pentabox itself is the only diagram that contributes on this cut, hence in this case $n = \Delta$. Thus,

$$n(12345; \ell_1, \ell_2) = \Delta + f_1(12345; \ell_1, \ell_2)\ell_1^2 + f_2(12345; \ell_1, \ell_2)(\ell_1 - p_1)^2 + \ldots.$$

- We determine the unknown rational functions f_i by considering the other two cuts, both of which are double–boxes.

A Double–Box Cut

- Take the same cut as previously leaving $(\ell_1 - p_1)^2$ nonzero.



$$\text{Cut}_{\text{double–box}} = \ell_1^2(\ell_1 - p_1 - p_2)^2 \ldots (\ell_1 + \ell_2)^2 \mathcal{A}^{\text{2–loop}}$$

$$= \frac{n(12345; \ell_1, \ell_2)}{(\ell_1 - p_1)^2} + \frac{n(12345; \ell_1, \ell_2) - n(21345; \ell_1, \ell_2)}{(p_1 + p_2)^2}$$

$$= \frac{\Delta(12345; \ell_1, \ell_2) + (\ell_1 - p_1)^2 f_2(12345; \ell_1, \ell_2)}{(\ell_1 - p_1)^2}$$

$$+ \frac{\Delta(12345; \ell_1, \ell_2) + (\ell_1 - p_1)^2 f_2(12345; \ell_1, \ell_2) - (p_1 \leftrightarrow p_2))}{(p_1 + p_2)^2}$$

Solving the Double–Box Cut

- We choose to express the cut in terms of irreducible numerators,

$$\text{Cut}_{\text{double–box}} = \frac{\Delta}{(\ell_1 - p_1)^2} + \Delta_2.$$

- The cut equation holds under arbitrary permutations of external momenta. So consider the same equation, taking $p_1 \leftrightarrow p_2$, and solve to obtain

$$f_2(12345; \ell_1, \ell_2) + f_2(21345; \ell_1, \ell_2)$$
$$= \Delta_2(12345; \ell_1, \ell_2) + \Delta_2(21345; \ell_1, \ell_2).$$

- This is a symmetry condition on f_2 and can be solved by taking $f_2 = \Delta_2 + g_2$, where $g_2(12345; \ell_1, \ell_2) = -g_2(21345; \ell_1, \ell_2)$.

Progressing to a Solution

- Once all the cut equations are solved, we are most of the way to a solution. For $\mathcal{N} = 4$ we also need to set diagrams containing triangles to zero.
- Ultimately we are left with a solution of the form

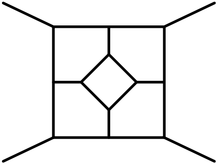
$$\begin{aligned} n(12345;\ell_1,\ell_2) = n^{\mathrm{CJ}}(12345;\ell_1,\ell_2) &- \chi(34512)\ell_1^2 \\ &- (\chi(13254) - \chi(25413))(\ell_1 - p_1)^2 \\ &+ (\chi(13254) - \chi(24513))(\ell_1 - p_1 - p_2)^2 \\ &- \chi(12345)(\ell_1 - p_1 - p_2 - p_3)^2, \end{aligned}$$

where n^{CJ} is a solution, first found by Carrasco & Johansson via method of ansatz (arXiv:1106.4711), and χ is a new function satisfying

$$\chi([1,2]345) = \chi(123[4,5]) = \chi(12345) - \chi(54321) = 0,$$
$$\chi(12345) + \chi(25341) + \chi(51342) = 0.$$

Outlook

- Our workflow can be summarised as:
  1. Evaluate cuts of planar integrands using unitarity methods.
  2. Use these to derive BCJ master numerators.
  3. From these extract the full amplitude using BCJ moves, and a corresponding gravity amplitude.
- The ansatz method has failed to produce a solution for the 5–loop, 4–point $\mathcal{N} = 4$ system,



- We would like to move beyond $\mathcal{N} = 4$, deriving numerators for pure YM amplitudes.

Thanks for listening!