Generalised Geometry & M-Theory

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Massless states in closed (bosonic) string theory:
metric g_{ab}, two-form B_{ab}, Dilaton ϕ

$$S_{gB} = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-h} \left(g_{ab}(X) \partial_\mu X^a \partial_\nu X^b h^{\mu\nu} + i B_{ab} \partial_\mu X^a \partial_\nu X^b \epsilon^{\mu\nu} \right)$$

$(\sqrt{-h}\epsilon^{12} = 1)$

Questions

- Does the B-field have a geometric interpretation?
- Can the theory be reformulated so that g, B appear on an equal footing?
- Is there a generalisation to M-Theory?

Outline

1. Introduction to Generalised Geometry
   - $T \oplus T^*$ Generalised Geometry
   - Generalised Metrics
2. Application to M-Theory
   - Idea
   - Simple Example: $n = 4$, Euclidean Signature
   - Higher Dimensions

Introduction to Generalised Geometry

Differential Geometry

- smooth manifold (M, D)
- smooth structure $D \Rightarrow$ tangent bundle T, Lie derivative/ bracket $\mathcal{L} : \Gamma(T) \times \Gamma(T) \to \Gamma(T)$
- Riemannian metric g: smoothly varying inner product on T
- Connection ∇ on T, (almost) complex structure on T, symplectic structure...

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Generalised Geometry

$T \rightarrow$ vector bundle $\mathcal{E} \xrightarrow{\pi} M$ with structure group G.
Simplest case: $T \rightarrow T \oplus T^*$

$T \oplus T^*$ Generalised Geometry

Let V be a vector space of dimension n, V^* dual space.
Consider $V \oplus V^*$.
Write $X + \xi \in V \oplus V^*,\ X \in V,\ \xi \in V^*$.

- Canonical symmetric bilinear form:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left(\xi(Y) + \eta(X) \right)$$

- $<,>$ has signature (n, n)

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$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left(\xi(Y) + \eta(X) \right)$$

- $<,>$ has signature (n, n)
- Define

$$O(V \oplus V^*) = \{ \Lambda \in \mathrm{Aut}(V \oplus V^*) | \langle \Lambda(x), \Lambda(y) \rangle = \langle x, y \rangle \; \forall x, y \in V \oplus V^* \} \cong O(n, n)$$

$$so(V \oplus V^*) = \{ T \in \mathrm{End}(V \oplus V^*) | \langle Tx, y \rangle + \langle x, Ty \rangle = 0 \; \forall x, y \in V \oplus V^* \}$$

Symmetries

Let $T \in so(V \oplus V^*)$. Then:

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}$$

- $A \in \text{End}(V)$
- $\beta : V^* \to V, \ \beta^* = -\beta$
- $B : V \to V^*, \ B^* = -B \ \rightsquigarrow \ B \in \wedge^2 V^* : \ B(X) = i_X B$

- *B*-**transform**

$$\exp\begin{pmatrix}0 & 0\\ B & 0\end{pmatrix} = \begin{pmatrix}\mathbb{1} & 0\\ B & \mathbb{1}\end{pmatrix}$$

Maps $X + \xi \mapsto X + \xi + i_X B$

- $GL(V)$-**action**

$$\exp\begin{pmatrix}A & 0\\ 0 & -A^*\end{pmatrix} = \begin{pmatrix}\exp(A) & 0\\ 0 & \exp(A^*)^{-1}\end{pmatrix}$$

$\rightsquigarrow$ Embedding $GL(n, \mathbb{R}) \hookrightarrow O(n, n)$

Let g be Riemannian Metric on M, X vector field, $X_p \neq 0$ for some $p \in M$.

$$\langle X + gX, X + gX \rangle = g(X, X) > 0 \text{ at } p.$$

Let $V = \text{span}(\{X + gX, X \in \Gamma(T)\})$. Then

$$T \oplus T^* = V \perp V^\perp, \text{ where } V^\perp = \text{span}(\{X - gX\})$$

Choice of Riemannian Metric

$\Rightarrow$ Choice of split $T \oplus T^* = V \perp V^\perp$, $<,>$ positive definite on V , dim V maximal.

Generalised Metrics

A *Generalised Metric* $\mathcal{H}$ on $T \oplus T^*$

is a smoothly varying family of *symmetric, positive definite* bilinear forms on $(T \oplus T^*)_p, \; p \in M$ which is compatible with the canonical inner product $<,> = \eta$ in the following way:

$$\eta^{-1}\mathcal{H}\eta^{-1} = \mathcal{H}^{-1}$$

Example:

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \quad g^* = g^{-1}$$

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Example 2:

$$\mathcal{H} = \begin{pmatrix} \mathbb{1} & 0 \\ -B & \mathbb{1} \end{pmatrix}^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -B & \mathbb{1} \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

$$\Rightarrow \mathcal{H}(X + \xi, Y + \eta) = g(X, Y) + g^*(\xi + i_X B, \eta + i_Y B)$$

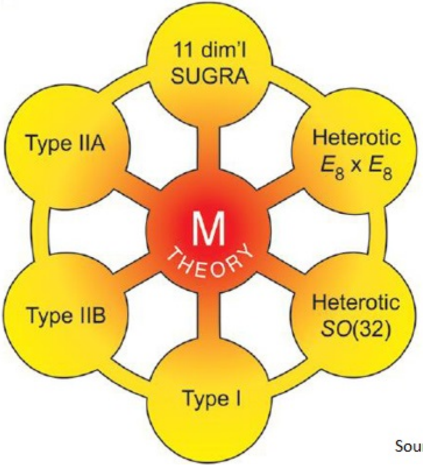
Generalised Metrics (2)

Choice of Generalised Metric

$\Leftrightarrow$ Choice of g Riemannian Metric & $B \in \Gamma(\wedge^2 T^*)$
$\Leftrightarrow$ Choice of split $T \oplus T^* = V \perp V^\perp$, $<,>$ positive definite on V, dim V maximal.

Physics: $B \rightsquigarrow$ 2-form gauge field; field strength $H = dB$.
$\Rightarrow$ Global structure of bundle & structure group modified
$\rightsquigarrow$ *generalised tangent bundle*

M-Theory



Source: www.ctc.cam.ac.uk

Field content of 11-dimentional supergravity:
metric g_{ab}, 3-form C_{abc}, Majorana gravitino ψ_a.

Aim: Build a corresponding Generalised Geometry

	generalised metric: g_{ab}, B_{ab}	generalised metric: g_{ab}, C_{abc}
$\Rightarrow$	Bundle: $T \oplus T^*$	Bundle: $T \oplus \wedge^2 T^*$

Simple example: $n = 4$, Euclidean Signature

Consider $\mathcal{E} = T \oplus \wedge^2 T^*$.

- $\dim(T \oplus \wedge^2 T^*)_p = 10 = 4 + 6$.
- Generalised vector:

$$U = v + \rho, U^I = \begin{pmatrix} v^i \\ \rho_{ij} \end{pmatrix}$$

Transformations

- $A \in SL(4),\ v^i + \rho_{jk} \mapsto A^i_j v^j + (A^{-1})^l_j (A^{-1})^m_k \rho_{lm}$ (15)
- $C \in \wedge^3 T^*,\ v^i + \rho_{jk} \mapsto v^i + \rho_{jk} + C_{ijk} v^i$ (4)
- $\gamma \in \wedge^3 T,\ v^i + \rho_{jk} \mapsto v^i + \frac{1}{2}\gamma^{ijk}\rho_{jk} + \rho_{jk}$ (4)
- Scale transformation with parameter $\alpha \in \mathbb{R}^\times$ (1)

Representation of $SL(5)$!

Generalised metric for $T \oplus \wedge^2 T^*$

Positive definite, symmetric bilinear form:

$$\mathcal{H}(v+\rho, w+\zeta) = g(v,w) + g^*(\rho + i_v C, \zeta + i_w C)$$

(g Riemannian metric, $C \in \wedge^3 T^*$)
where

$$(g^*)^{abcd} = \frac{1}{2}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right).$$

$$\begin{aligned}\dim(\mathrm{Sym}_4) + \dim(\wedge^3 T^*)_p &= 10 + 4 = 14 \\ &= 24 - 10 = \dim(SL(5)/SO(5))\end{aligned}$$

Higher Dimensions

d	n	G	H
10	1	$SO(1,1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
9	2	$SL(2,\mathbb{R}) \times \mathbb{R}$	$SO(2)$
8	3	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$SL(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5)$	$S(O(5) \times O(5))$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

Dimensional reduction of $d = 11$ supergravity
$\mathbb{R}^{10,1} \to \mathbb{R}^{d-1,1} \times T^n$

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10	1	$SO(1,1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
9	2	$SL(2,\mathbb{R}) \times \mathbb{R}$	$SO(2)$
8	3	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$SL(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5)$	$S(O(5) \times O(5))$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

Conclusion

Generalised Geometry

allows a formulation of string theory and M-Theory in which the massless bosonic fields naturally arise from the same geometric object.

Bibliography

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Extended Generalised Geometries

- $\mathcal{E} \xrightarrow{\pi} M$ vector bundle with (non-compact) structure group G.
- $H \subset G$ maximal compact subroup
  $\Rightarrow \ \mathcal{E} \rightsquigarrow \bar{\mathcal{E}}$ with structure group H.
  Example: $G = O(n,n),\ H = O(n) \times O(n)$.
  $\leftrightarrow T \oplus T^* = V \perp V^\perp$
- $\bar{\mathcal{E}} \leftrightarrow$ choice of Generalised Metric; parametrised by G/H on each fibre.
  H-invariant metric $\bar{\mathcal{H}}$ on $\bar{\mathcal{E}} \ \rightsquigarrow$ generalised metric on $\mathcal{E}$:

$$s \in \Gamma(\mathcal{E}),\ \mathcal{V} : \mathcal{E} \to \bar{\mathcal{E}},\ \mathcal{H}(s,s) = \bar{\mathcal{H}}(\mathcal{V}s, \mathcal{V}s)$$

Representation theory of $SL(5)$

Two-forms $\wedge^2\mathbb{R}^5$**:** dim $= (5^2-5)/2 = 10$
Split $SL(5) \supset SL(4):\ 10 = 6 + 4$

$$\wedge^2\mathbb{R}^5 \ni B_{5\times5} = \left(\begin{array}{c|c} B_{4\times4} & v_4 \\ \in \wedge^2\mathbb{R}^4 & \in \mathbb{R}^4 \\ \hline -v_4^T \in \mathbb{R}^4 & 0 \end{array}\right)$$

Adjoint Representation: dim $= 5^2 - 1 = 24$
Split $SL(5) \supset SL(4):\ 24 = 15 + 4 + 4 + 1$

$$sl(5) \ni A_{5\times5} = \left(\begin{array}{c|c} A_{4\times4} & v_4 \\ \in sl(4) & \in \mathbb{R}^4 \\ \hline w_4 \in \mathbb{R}^4 & 0 \end{array}\right) + \left(\begin{array}{c|c} -\alpha/4 \cdot \mathbb{1} & 0 \\ \hline 0 & \alpha \end{array}\right)$$

$\Rightarrow\ T \oplus \wedge^2 T^*$ has structure group $SL(5)$ via the 10-dimensional representation!