BCJ duality, the Double copy and Black Holes

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IT'S
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KEEP
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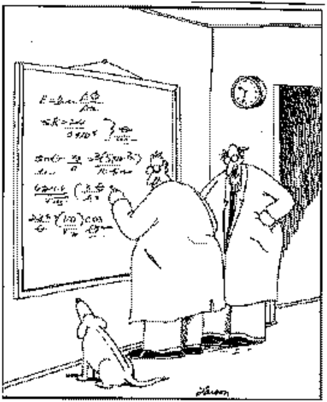
theory.png 

Figure : They act so cute when they try to understand Quantum Field Theory.

- BCJ duality is a kinematic identity for n-point tree level color-ordered gauge theory amplitudes,

$$\mathcal{A}_n^{tree}(1,2,...,n) = g^{n-2} \sum_{\mathcal{P}(2,3,\ldots,n)} Tr[T^{a_1} T^{a_2} \cdots T^{a_n}] A_n^{tree}(1,2,...,n) \quad (1)$$

- Kinematic analog of Jacobi identity for numerators in the amplitudes.

- Using generalized unitarity, the numerators identity has applicatons at higher loops.

- Color-ordered, tree level amplitudes satisfy some identities (cyclic, reflection and photon-decoupling). At four points, photon-decoupling identity reads,

$$A_4^{tree}(1,2,3,4) + A_4^{tree}(1,3,4,2) + A_4^{tree}(1,4,2,3) = 0. \quad (2)$$

- Then, using kinematic considerations we obtain the following relations between four point amplitudes,

$$\begin{aligned} tA_4^{tree}(1,2,3,4) &= uA_4^{tree}(1,3,4,2), \\ tA_4^{tree}(1,4,2,3) &= sA_4^{tree}(1,3,4,2), \\ sA_4^{tree}(1,2,3,4) &= uA_4^{tree}(1,4,2,3), \end{aligned} \quad (3)$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$, $u = (k_1 + k_3)^2$.

- Expressing these tree color-ordered amplitudes in terms of the poles that appear,

$$\begin{aligned} A_4^{tree}(1,2,3,4) &\equiv \frac{n_s}{s} + \frac{n_t}{t}, \\ A_4^{tree}(1,3,4,2) &\equiv -\frac{n_u}{u} - \frac{n_s}{s}, \\ A_4^{tree}(1,4,2,3) &\equiv -\frac{n_t}{t} + \frac{n_u}{u}. \end{aligned} \tag{4}$$

- Comparing the last two expressions, we get the relation,

$$n_u = n_s - n_t, \tag{5}$$

which mimics the Jacobi identity,

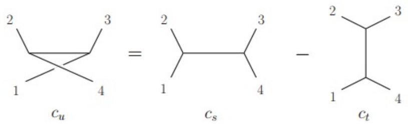


FIG. 2: The Jacobi identity relating the color factors of the u, s, t channel "color diagrams". The color factors are given by dressing each vertex with an $\tilde{f}^{abc}$ following a clockwise ordering.

$$c_u = c_s - c_t, \tag{6}$$

where,

$$c_u \equiv \tilde{f}^{a_4 a_2 b} \tilde{f}^{b a_3 a_1}, \; c_s \equiv \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \; c_t \equiv \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}. \tag{7}$$

- Given three dependent color factors c_α, c_β, c_γ associated with tree level color diagrams, scattering amplitudes can be decomposed into kinematic diagrams with numerator factors n_α, n_β, n_γ that satisfy

$$c_\alpha - c_\beta + c_\gamma = 0, \Rightarrow \; n_\alpha - n_\beta + n_\gamma = 0. \tag{8}$$

- For example, in the five-point case, the diagrams in the figure satisfy the color identity

$$c_3 = c_5 - c_8, \tag{9}$$

where

$$c_3 \equiv \tilde{f}^{a_3 a_4 b} \tilde{f}^{b a_5 c} \tilde{f}^{c a_1 a_2}, \; c_5 \equiv \tilde{f}^{a_3 a_4 b} \tilde{f}^{b a_2 c} \tilde{f}^{c a_1 a_5}, \; c_8 \equiv \tilde{f}^{a_3 a_4 b} \tilde{f}^{b a_1 c} \tilde{f}^{c a_2 a_5}. \tag{10}$$

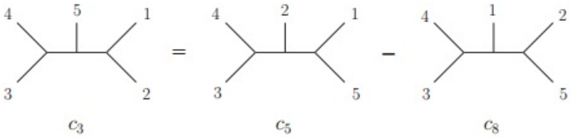


FIG. 4: The Jacobi identity at five points. These diagrams can be interpreted as relations for color factors, where each color factor is obtained by dressing the diagrams with $\tilde{f}^{abc}$ at each vertex in a clockwise ordering. Alternatively it can be interpreted as relations between the kinematic numerator factors of corresponding diagrams, where the diagrams are nontrivially rearranged compared to Feynman diagrams.

- Then, it is possible to write the numerators, in such a form that they satisfy the same identities as the color factors. This is,

$$c_3 - c_5 + c_8 = 0, \Rightarrow\ n_3 - n_5 + n_8 = 0, \tag{11}$$

where the kinematic numerators come from expressing the full color dressed amplitude via

$$\mathcal{A}_5^{tree} = g^3 \sum_{i=1}^{15} \frac{n_i c_i}{p_i}. \tag{12}$$

- This will have as a consequence, simple relations between color-ordered amplitudes. For example

$$A_5^{tree}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{tree}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{tree}(1,4,3,2,5)}{s_{13}s_{24}} \quad (13)$$

(and another three of those).

KLT relations

- Derived by Kawai, Lewellen and Tye in 1986.
- First uncovered in string theory, hold in field theory (string's low energy limit).
- Relate gauge and gravity theories amplitudes. For example,

$$M_5^{tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{tree}(1,2,3,4,5)\tilde{A}_5^{tree}(2,1,4,3,5) + i s_{13} s_{24} A_5^{tree}(1,3,2,4,5)\tilde{A}_5^{tree}(3,1,4,2,5). \tag{14}$$

- BCJ conjectured this duality is true to all loop orders and (partially inspired by KLT relations) we can write gravity theories scattering amplitudes by "squaring" a gauge theory scattering amplitude. This process is called Double copy.

- A general massless m-point gauge theory amplitude in d space-time can be written as,

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{\ell=1}^{L} \frac{d^d p_\ell}{(2\pi)^d} \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \qquad (15)$$

- If kinematic numerators satisfy BCJ relations, the m-point, L-loop gravity amplitude will be,

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{\ell=1}^{L} \frac{d^d p_\ell}{(2\pi)^d} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \qquad (16)$$

Why solutions?

- Trying to understand better the origin of BCJ and Double copy.
- Because they are defined in a purely perturbative context, multiloop calculations make difficult to explorate the deeper meaning.
- Do features manifest themselves in a clasical context? (Or at Lagrangian level).

- In Kerr-Schild coordinates, spacetime metric may be written in the form,

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ &= \eta_{\mu\nu} + k_\mu k_\nu \phi, \end{aligned} \tag{17}$$

where the vector k_μ has the property of being null with both the Minkowski and the Kerr-Schild metrics:

$$\eta_{\mu\nu} k_\mu k_\nu = 0 = g_{\mu\nu} k_\mu k_\nu. \tag{18}$$

- In terms of function ϕ and vector k_μ, one has the tensor,

$$R^\mu_\nu = \frac{1}{2}\left(\partial^\mu\partial_\alpha(\phi k^\alpha k_\nu) + \partial_\nu\partial_\alpha(\phi k^\alpha k^\mu) - \partial^2(\phi k^\mu k_\nu)\right). \quad (19)$$

- In the stationary case (where $\partial_0 = 0$, $k^0 = 1$), Einstein vacuum equations are,

$$R^0_0 = \frac{1}{2}\nabla^2\phi \quad (20)$$

$$R^i_0 = -\frac{1}{2}\partial_j\left[\partial^i(\phi k^j) - \partial^j(\phi k^i)\right] \quad (21)$$

$$R^i_j = \frac{1}{2}\partial_l\left[\partial^i(\phi k^l k_j) + \partial_j(\phi k^l k^i) - \partial^l(\phi k^i k_j)\right]. \quad (22)$$

- If we define a vector field $A_\mu = \phi k_\mu$, the Einstein vacuum equations $R_{\mu\nu} = 0$ imply, in the stationary case,

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu(\phi k^\nu) - \partial^\nu(\phi k^\mu)) = 0. \tag{23}$$

- Let,

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi, \tag{24}$$

be a stationary solution of the Einstein equations, then,

$$A^a_\mu = c_a \phi k^\mu, \tag{25}$$

is a solution of the Yang Mills equations. This constitutes a class of solutions identifiable between gauge and gravity theories.

- The Gauge solution is refered as single copy, or square root of the gravity solution.

- Most general spherically symmetric solution of vacuum Einstein equation.
- Considering the energy-momentum tensor,

$$T^{\mu\nu} = M v^{\mu} v^{\nu} \delta^{(3)}(\mathbf{x}), \tag{26}$$

where $v^{\mu} = (1, 0, 0, 0)$. The exterior metric may be put in the form,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r} k_{\mu} k_{\nu}, \tag{27}$$

(which is in Kerr-Schild form), where,

$$k^{\mu} = \left(1, \frac{x^i}{r}\right), \; r^2 = x^i x_i, \; 1 = 1...3. \tag{28}$$

- Using $\kappa^2 = 16\pi G$, the graviton will be,

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu,\ \phi = \frac{M}{4\pi r}. \tag{29}$$

And we can have the single copy,

$$A^\mu = \frac{g c_a T^a}{4\pi r} k_\mu, \tag{30}$$

via the replacements,

$$\frac{\kappa}{2} \to g,\ M \to c_a T^a,\ k_\mu k_\nu \to k_\mu,\ \frac{1}{4\pi r} \to \frac{1}{4\pi r}. \tag{31}$$

- Given that this is a solution of Abelian Maxwell equations, we can perform a gauge transformation,

$$A^a_\mu \to A^a_\mu + \partial_\mu \chi^a(x), \tag{32}$$

Let us choose,

$$\chi^a = -\frac{gc_a}{4\pi} \log\left(\frac{r}{r_0}\right). \tag{33}$$

In this gauge, one has,

$$A_\mu = \left(\frac{gc_a T^a}{4\pi r}, 0, 0, 0\right). \tag{34}$$

This is a Coulomb-like solution.

- The uncharged, rotating black hole (Kerr) can be put in Kerr-Schild form, with the graviton,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi(r) k_\mu k_\nu, \tag{35}$$

where,

$$\phi(r) = \frac{2MGr^3}{r^4 + a^2 z^2}, \tag{36}$$

and,

$$k^\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right), \tag{37}$$

and r is implicitly defined by,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{38}$$

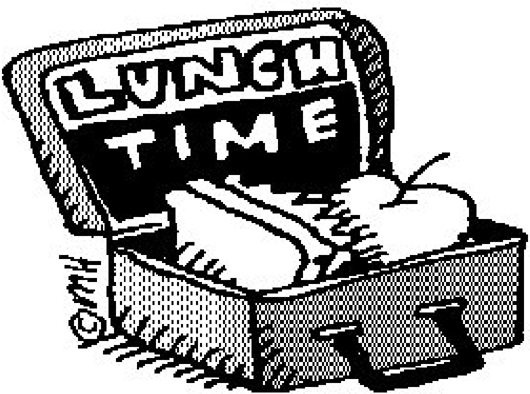
- Following the Kerr-Schild single copy procedure, one may construct the gauge field,

$$A^a_\mu = \frac{g}{4\pi}\phi(r) c_a k\mu, \tag{39}$$

where again this is a solution to the Abelian Maxwell equations.

This single copy procedure, can be applied to time dependent solutions, like,

- Plane waves solutions.
- Shockwave solutions.
- Taub-NUT solutions. (?)(Further Work)



Thank you.